Estimating the number of ergodic measures of a minimal subshift knowing the geometry of its Rauzy graphs

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Abstract

As a geometric implementation of Boshernitzan's methods, we give some bounds on the number of ergodic invariant measures of a minimal subshift, that depend on the geometry of two combinatorial objects associated to its language: the Rauzy graphs and the tree of left special factors.

Keywords: symbolic dynamics, subshift, simplex of invariant measures, unique ergodicity, Rauzy graph.

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PROVISIONAL VERSION TO BE ENLARGED AND REORGANISED.

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Introduction

Symbolic dynamics naturally appear as codings of dynamical systems with finite entropy: given a suitable finite partition $\mathcal{P}$ of the underlying space X of such a system, we can follow the atoms of the partition encountered by each orbit to create infinite words on the alphabet $\mathcal{P}$. Whenever the partition is well chosen, this establishes a "dictionary" between the dynamical system and a subshift that helps to find computable informations. For ergodic measurable dynamical systems, we can always get an isomorphism between the system and a subshift [Kri]. For topological dynamical systems, there are some obstructions since any nontrivial subshift is an expansive map on a Cantor space (this seldom corresponds to what we consider as a geometrical dynamical system). Anyway, a topological semi-conjugacy can be obtained for particular geometrical dynamical systems, like interval exchange transformations [Kea] or pseudo-Anosov homeomorphisms on a compact surface (the measurable Markov partitions constructed in [FaLaPo] are studied in the topological context in [Adl]). This is usually sufficient to preserve most of the dynamical properties, like mixing, entropy, unique ergodicity...

We are interested here in topological dynamics of minimal subshifts and especially in the estimation of the number of their ergodic invariant measures. In that direction, Grillenberger constructed uniquely ergodic minimal subshifts of any given entropy [Gri].

For null entropic ones, complexity refines the notion of entropy. Boshernitzan gave a bound on the number of ergodic invariant measures for subshifts of linear complexity [Bos1]. He also gave a sufficient condition for such systems to be uniquely ergodic [Bos3] that leads to an elegant symbolic proof of the well known theorem of Masur [Mas] and Veech [Vee2] about interval exchange transformations. In some such calm flows, dynamical properties appear to depend highly on dipohantine approximation (see for example [Rau2] [Tij] [Dur] [Che] [McM] [BeFeZa]) and precise combinatorial tools have been used to keep track of those.
This paper is situated somewhere around there, and tries to understand possible relations between those combinatorial objects and the simplex of invariant measure of a minimal subshift.

The paper is organized as follows: in section 1, we give some basic definitions and introduce the simplex of invariant measures as well as two combinatorial objects associated to a minimal subshift that are transverse to eachother: the Rauzy graphs and the tree of left special factors. Section 2 is devoted to give a bound of the number of ergodic invariant measures that depends on the geometry of the Rauzy graphs [Rau1]: we prove the following theorem and give some applications.

Theorem 1. *A K-deconnectable minimal subshift has at most K S-invariant ergodic measures.*

See section 2 for a definition of K-deconnectable. Section 3 explains the limits of such an approach. We discuss the evolution of Rauzy graphs and synchronization problems that appear in their study. Section 4 tries to solve this and provides a bound based on the tree of left special factors:

Theorem 3. *Let (X, S) be a minimal subshift of linear complexity such that there exists a positive integer K and a sequence (f_n) of cuts such that*

- $h(f_n) \xrightarrow[n\to\infty]{} \infty$
- $osc(f_n) \in o(h(f_n))$
- *a representation of $T(X)$ meets the graph of f_n at at most K points ($n \in \mathbb{N}$).*

Then (X, S) admits at most K ergodic invariant measures.

See sections 1.3 and 4 for definitions.

1 Landscape

Let A be a finite set (called an *alphabet*), endowed with the discrete topology.
We note $S \stackrel{\text{def}}{=} \begin{pmatrix} A^{\mathbb{N}} & \longrightarrow & A^{\mathbb{N}} \\ x = x_0x_1\dots x_n\dots & \longmapsto & x_1x_2\dots x_{n+1}\dots \end{pmatrix}$ for the shift.
Let X be a nonempty closed subset of $A^{\mathbb{N}}$ (endowed with the product topology) stable under S and that is minimal for those properties: we say that (X, S) is a *minimal subshift* (we still note S for $S|_X$). We assume moreover that (X, S) is aperiodic i.e. X is infinite.

1.1 Invariant measures

A way to describe a minimal subshift (X, S) or more generally a topological dynamical system, is to study the set $\mathcal{M}(X, S)$ of Borel probability measures on X that are invariant under S. Thanks to Riesz representation, this set can be identified to a nonempty compact convex subset of $C^0(X, \mathbb{R})'$ endowed with the weak-star topology.

A S-invariant measure $\mu \in \mathcal{M}(X, S)$ is said to be *ergodic* if the only Borel sets $A \subset X$ such that $S^{-1}(A) = A$ have measure $\mu(A) = 0$ or 1. Such a measure satisfies Birkhoff's theorem:

$$\forall f \in L^1(X, \mathbb{R}) \qquad \frac{1}{n}\sum_{k=0}^{n-1} f \circ S^k \xrightarrow[n\to\infty]{\mu-a.e.} \int_X f d\mu$$

Let $\mathcal{E}(X, S)$ denote the set of ergodic invariant measures: it is the set of extremal points of $\mathcal{M}(X, S)$ and it is therefore nonempty. Two different elements of $\mathcal{E}(X, S)$ are mutually singular.

A minimal subshift is said to be *uniquely ergodic* if $\text{card}(\mathcal{M}(X, S)) = 1$. One advantage of such a situation is that, the unique invariant measure μ is ergodic, moreover the convergence in Birkhoff's theorem is uniform for continuous functions.

For more details on this subsection, see [DeGrSi].

1.2 Rauzy graphs

A way to grasp a minimal subshift (X, S) is to look at the finite words that appear as subwords of some word x of X. A *finite word* is an element u of A^n, for some integer n that is called the *length* of u and is denoted by $l(u)$.
If x is a word (finite or not) and $0 \leq i \leq j$, then $x_{i \to j}$ denotes $x_i x_{i+1} x_{i+2} \dots x_j$.
If u and v are two finite words, we note $\#(u, v)$ the number of occurrences of u in v:

$$\#(u,v) \stackrel{\text{def}}{=} \text{card}\{k \leq l(v) - l(u) \mid u_{0 \to l(u)-1} = v_{k \to k+l(u)-1}\}$$

If $u = u_0 \dots u_{n-1} \in A^n$ is a finite word, we can define the *cylinder*

$$[u] \stackrel{\text{def}}{=} \{x \in X \mid (\forall i \leq n-1)(x_i = u_i)\}$$

The cylinders are clopen sets and form a basis of the topology of X.
For any integer n, $L_n(X)$ denotes the set of finite words of length n occurring in some (equivalently, any) infinite word x of X.

$$L_n(X) \stackrel{\text{def}}{=} \{u \in A^n \mid [u] \neq \emptyset\}$$

Any word in $L_{n+1}(X)$ is naturally linked to two words in $L_n(X)$ (prefix and suffix).
A way to keep track of this factorial structure of $L(X) \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} L_n(X)$ is the use of Rauzy graphs (see [Rau1] [Bos2]). For any integer n, we define the n^{th} *Rauzy graph* $G_n(X)$ as follows :

- the set of vertices of $G_n(X)$ is $L_n(X)$.
- there is an (oriented) edge from u to v in $G_n(X)$ if there exists w in $L_{n+1}(X)$ such that w begins with u and ends with v.

The knowledge of all $G_n(X)$ allows to reconstruct X in a profinite manner.

1.3 The tree of left special factors

A word u in $L(X)$ is called a *left special factor* if there exist two different letters a and b in A such that au and bu are in $L(X)$. Let $LS_n(X)$ denote the set of left special factors of length n. Viewed in the Rauzy graph, left special factors of length n are the vertices of $G_n(X)$ with incoming degree greater than 1. We can define right special factors in the same way.
Since any prefix of a left special factor is still a left special factor, they form a tree $T(X)$ whose root is the empty word ϵ:

- the vertices of $T(X)$ are the elements of $LS(X) \stackrel{\text{def}}{=} \bigcup_{n \geq 0} LS_n(X)$.
- there is an edge from u to v if u is a prefix of v and $l(u) = l(v) - 1$.

Such a tree can be embedded in $[0, 1] \times \mathbb{R}_+$ in such a way that vertices in $LS_n(X)$ have second coordinate n and edges are non overlapping segments. Such a representation of $T(X)$ will still be denoted by $T(X)$ when there is no possible confusion.

For more details on Rauzy graphs and the tree of left special factors, see [Cas].

2 A clean version

We will provide a bound of $\mathrm{card}(\mathcal{E}(X,S))$ that depends on the fragility of the $G_n(X)$:
If $K \geq 1$, a minimal subshift (X,S) is said to be *K-deconnectable* if there exists an extraction (i.e. a strictly increasing integer sequence) $\alpha \in \uparrow(\mathbb{N},\mathbb{N})$ and a constant $K' \geq 1$ such that for all $n \geq 1$ there exists a subset $D_{\alpha(n)} \subset L_{\alpha(n)}(X)$ of at most K vertices such that every path in $G_{\alpha(n)}(X) \setminus D_{\alpha(n)}$ is of length less than $K'\alpha(n)$ (in particular it does not contain any cycle).
This means that we can disconnect (in a specific way) infinitely many Rauzy graphs by removing at most K vertices.

Theorem 1. *A K-deconnectable minimal subshift has at most K S-invariant ergodic probability measures.*

Proof: We will first construct at most K possible candidates and then prove that they are the only ones.
Step 1: *We construct the candidates to be the only ergodic invariant probability measures.*
For any integer n, let $d_{1,\alpha(n)}, d_{2,\alpha(n)}, \ldots, d_{K,\alpha(n)}$ be an enumeration of $D_{\alpha(n)}$ (there is no loss of generality to consider that all the $D_{\alpha(n)}$ have exactly K elements).
For this, we approximate X from the outside by K sequences of periodic subshifts as follows: for $i \leq K$ and $n \in \mathbb{N}$, let

$$\mu_{i,\alpha(n)} \stackrel{\mathrm{def}}{=} \frac{1}{\alpha(n)} \sum_{k=0}^{\alpha(n)-1} \delta_{S^k(d^{\omega}_{i,\alpha(n)})}$$

($d^{\omega}_{i,\alpha(n)}$ denotes $d_{i,\alpha(n)}d_{i,\alpha(n)}d_{i,\alpha(n)}\ldots$ and δ stands for the one-point Dirac's measure).
$\mu_{i,\alpha(n)}$ is the only element of $\mathcal{M}(A^{\mathbb{N}},S)$ that gives measure 1 to the periodic subshift generated by the periodic word $d^{\omega}_{i,\alpha(n)}$.
By compacity of $\mathcal{M}(A^{\mathbb{N}},S)^K$, there exists an extraction β such that for each $i \leq K$,

$$\mu_{i,\alpha\circ\beta(n)} \xrightarrow[n\to\infty]{} \mu_i$$

for some μ_i in $\mathcal{M}(A^{\mathbb{N}},S)$.
Note that if X is aperiodic, the $\mu_{i,\alpha(n)}$ give measure 0 to X. Anyway,

Step 2: *We show that for $i \leq K$, $\mu_i(X) = 1$.*
Since X is closed in $A^{\mathbb{N}}$, we have the following approximation by open sets:

$$X = \overline{X} = \bigcap_{n\in\mathbb{N}} \bigcup_{u\in L_n(X)} [u]$$

($[u]$ is considered here as a a cylinder in the current dynamical system: $A^{\mathbb{N}}$).
Let $n \geq 1$ and $k \geq n$. For $i \in \{0,\ldots,\alpha\circ\beta(k)-n\}$, $(d^{\omega}_{i,\alpha\circ\beta(k)})_{i\to i+n-1} \in L_n(X)$ (it is a subword of $d_{i,\alpha\circ\beta(k)}$). Hence $\mu_{i,\alpha\circ\beta(k)}(\bigcup_{u\in L_n(X)}[u]) \geq (\alpha\circ\beta(k)-n+1)/\alpha\circ\beta(k)$.
Letting k tend to infinity, since the characteristic function of $\bigcup_{u\in L_n(X)}[u]$ is continuous, we have $\mu_i(\bigcup_{u\in L_n(X)}[u]) = 1$. By countable intersection (n is arbitrary), we have $\mu_i(X) = 1$.

Hence, we can still denote by μ_i for the restriction of μ_i to X.

Step 3: *Let μ be an ergodic measure on X. We show that μ is one of the μ_i.*
By Birkhoff's theorem, there is x in X such that for any u in $L(X)$,

$$\mu([u]) = \lim_{n\to\infty} \frac{1}{n}\#(u, x_{0\to n+l(u)-2}) = \lim_{n\to\infty} \frac{1}{n}\#(u, x_{0\to n-1})$$

Let n be a fixed positive interger. We decompose x into blocks of length $(K'+1)\alpha\circ\beta(n)$: $x = B_0.B_1.B_2.B_3.B_4\dots$ with $B_j = x_{(K'+1)\alpha\circ\beta(n)j\to(K'+1)\alpha\circ\beta(n)(j+1)-1}$. By hypothesis, any B_j contains an occurrence of one of the $d_{i,\alpha\circ\beta(n)}$ (B_j can be viewed as a path of length $K'\alpha\circ\beta(n)$ in $G_{\alpha\circ\beta(n)}(X)$). So there exists some $i_{\alpha\circ\beta(n)}$ such that the upper density of the set $\{j \in \mathbb{N} \mid \#(d_{i_{\alpha\circ\beta(n)},\alpha\circ\beta(n)}, B_j) \geq 1\}$ is at least $1/K$. Let γ be an extraction such that $i_{\alpha\circ\beta\circ\gamma(.)}$ is constant with value denoted by i. We note $\tilde{\alpha}$ for $\alpha\circ\beta\circ\gamma$

We will show that $\mu = \mu_i$. Let u be a finite word in $L(X)$. Let n be an integer greater than $l(u)$. There is an extraction δ such that for any integer m,

$$\mathrm{card}\{j < \delta(m) \mid \#(d_{i,\tilde{\alpha}(n)}, B_j) \geq 1\} \geq \delta(m)/2K$$

Therefore

$$\#(u, B_0.B_1...B_{\delta(m)-1}) \geq \frac{\delta(m)}{2K}\#(u, d_{i,\tilde{\alpha}(n)})$$

Hence

$$\mu([u]) = \lim_{m\to\infty} \frac{1}{(K'+1)\tilde{\alpha}(n)\delta(m)}\#(u, B_0.B_1...B_{\delta(m)-1}) \geq \frac{1}{2K(K'+1)\tilde{\alpha}(n)}\#(u, d_{i,\tilde{\alpha}(n)})$$

Moreover, we can control the frequency of occurrences of u in $d^{\omega}_{i,\tilde{\alpha}(n)}$ by counting separately the occurrences of u that fall in some $d_{i,\tilde{\alpha}(n)}$ and the occurrences of u that appear bettween two consecutive occurences of $d_{i,\tilde{\alpha}(n)}$ (boundary effect):

$$\mu_{i,\tilde{\alpha}(n)}([u]) = \frac{1}{\tilde{\alpha}(n)}\sum_{k=0}^{\tilde{\alpha}(n)-1} \delta_{S^k(d^{\omega}_{i,\tilde{\alpha}(n)})}([u]) \leq \frac{1}{\tilde{\alpha}(n)}(\#(u, d_{i,\tilde{\alpha}(n)}) + l(u))$$

Therefore,

$$\mu_{i,\tilde{\alpha}(n)}([u]) \leq \frac{1}{\tilde{\alpha}(n)}\#(u, d_{i,\tilde{\alpha}(n)}) + \frac{l(u)}{\tilde{\alpha}(n)} \leq 2K(K'+1)\mu([u]) + \frac{l(u)}{\tilde{\alpha}(n)}$$

Letting n tend to infinity, we have $\mu_i([u]) \leq 2K(K'+1)\mu([u])$, so μ_i is absolutely continuous relatively to μ.

Since μ_i is S-invariant and μ is ergodic, we have $\mu_i = \mu$ and there are at most K S-invariant ergodic measures.

$\square$

We will now see how this result can be effectively used.

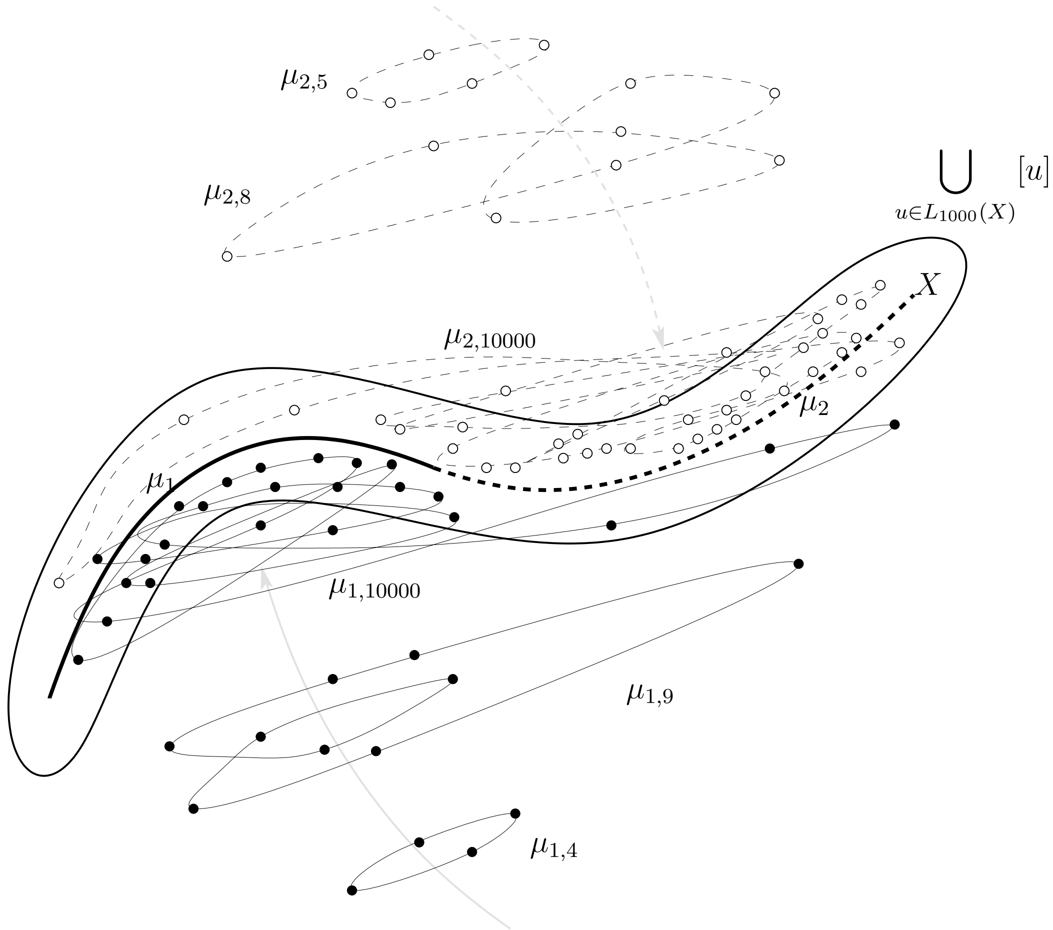


Figure 1: Approximation of a 2-deconnectable subshift with two ergodic measures by periodic orbits.

2.1 Subshifts with linear complexity

Let X be a minimal subshift. We define the *complexity* of X as the sequence defined by $p_n(X) \stackrel{\text{def}}{=} \text{card}(L_n(X))$. X is said to be of *linear complexity* if there exists an integer K such that

$$\limsup_{n\to\infty} \frac{p_n(X)}{n} \leq K$$

Since $p_{n+1}(X)$ counts the edges of $G_n(X)$ and $p_n(X)$ counts the vertices of $G_n(X)$, we have the majoration:

$$\text{card}(LS_n(X)) \leq p_{n+1}(X) - p_n(X)$$

We have the following result

Theorem 2. *[Bos1]*
Let (X, S) be a minimal subshift such that

$$\liminf_{n\to\infty} \frac{p_n(X)}{n} \leq K$$

then X admits at most K ergodic invariant measures.

Proof: There is an extraction $\alpha \in\uparrow(\mathbb{N},\mathbb{N})$ such that for all integer n, $p_{\alpha(n)}(X) \leq (K+1)\alpha(n)$ and $p_{\alpha(n)+1}(X) - p_{\alpha(n)}(X) \leq K$. For n in $\mathbb{N}$, let $D_{\alpha(n)} = LS_n(X)$, whose cardinal is not greater than K.
Let $n \in \mathbb{N}$. Any loop $\mathcal{O}$ in $G_{\alpha(n)}(X)$ must contain a left special factor: since X is aperiodic, there exists a finite word u in $L_n(X) \setminus \mathcal{O}$, and since X is minimal, there exists a path from u to any vertex of $\mathcal{O}$, so the first vertex of $\mathcal{O}$ that this path meets is a left special factor. Therefore, $G_{\alpha(n)}(X) \setminus D_{\alpha(n)}$ does not contain any loop.
So, a path in $G_{\alpha(n)}(X) \setminus D_{\alpha(n)}$ is necessarilly injective and cannot be of length greater than $\mathrm{card}(L_{\alpha(n)}(X)) \leq (K+1)\alpha(n)$.

□

Hence, theorem 1 can be understood as a geometric interpretation of Boshernitzan's result. Anyway, it is also fruitful for subshift that are not of linear complexity.

2.2 Arnoux-Rauzy subshifts

A minimal subshift (X, S) on an alphabet A is said to be *Arnoux-Rauzy* [ArnRau] if for any n in $\mathbb{N}$:

- $p_n(X) = (\mathrm{card}(A) - 1)n + 1$.
- $G_n(X)$ admits exactly one left special factor and one right special factor.

In particular we get the following proposition for free:

Proposition 1. *A minimal Arnoux-Rauzy subshift is* 1*-deconnectable and therefore uniquely ergodic.*

2.3 Multi-scale quasiperiodic subshifts

Let x be an infinite word on A. A finite word q is said to be a *quasiperiod* of x if x is covered by the occurences of q (in particular, q is a prefix of x) [Mar]. A minimal subshift (X, S) is said to be *multi-scale quasiperiodic* if it contains an element x that admits infinitely many quasiperiods [MoMa].

Proposition 2. *A multi-scale quasiperiodic subshift X is* 1*-deconnectable, and therefore uniquely ergodic.*

Proof: Let x be an element of X that admits infinitely many quasiperiods, and let q be a quasiperiod of x. For each u in $L_{l(q)}(X)$, let $\varphi(u)$ denotes the length of the longest prefix of u that is also a suffix of q. If $u \to v$ in $G_{l(q)}(X)$, then either $v = q$ or $1 \leq \varphi(v) < \varphi(u) \leq l(q)$.
Hence, $G_{l(q)}(X) \setminus \{q\}$ does not contain any path of length greater than $l(q)$, and since we cas reproduce this for infinitely many q, X is 1-deconnectable.

□

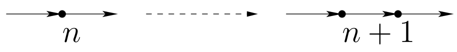
We have to notice that there exist multi-scale quasiperiodic subshifts that are *not* of linear complexity:

Proposition 3. *[MoMa] For each positive function $f : \mathbb{N} \to \mathbb{R}_+^*$ that converges to zero, there exists a multi-scale quasiperiodic subshift (X, S) such that $\frac{1}{n}\log(p_n(X)) \geq f(n)$ for infinitely many n. In particular, we can ask to $p_n(X)$ to grow faster than any polynomial on a subsequence.*

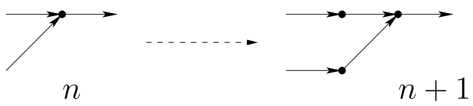
3 Transition-discussion: evolution of Rauzy graphs

To introduce the problematic of synchronization, let us see how do Rauzy graphs evolve with n. The following analysis is due to Rauzy [ArnRau] for bursting and splitting and Cassaigne [Cas] for the description of bispecial factors. For convenience, we restrict to the case where $\mathrm{card}(A) = 2$, therefore incoming and outgoing degree are not bigger than two. The vertices of $G_{n+1}(X)$ correspond to the edges of $G_n(X)$ and the edges of $G_{n+1}(X)$ will be described now. Let us look locally around the vertices of $G_n(X)$:

Simple extension A word in $L_n(X)$ that is not a special factor has incoming and outgoing degree 1, so this will lead in $G_{n+1}(X)$ to two vertices connected with an edge.



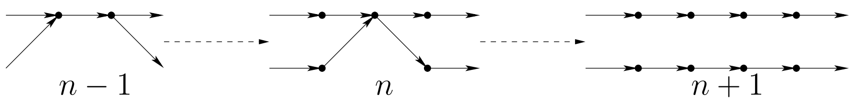
Splitting of a common special factor A left special factor that is not a right special factor has incoming degree 2 and outgoing degree 1, so this will lead in $G_{n+1}(X)$ to two vertices connected to a third one with two edges. We can understand this phenomenon as a slit of $G_n(X)$.



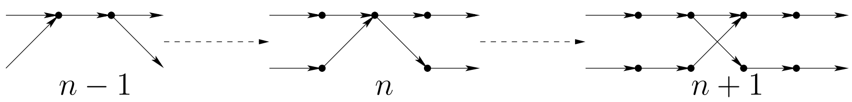
The same phenomenon appears for right special factors that are not left special factors. Hence a path between such a left and a right factor becomes shorter and shorter, until a...

Bursting of a bispecial factor A *bispecial factor* is a word that is both left and right special factor: it has incoming and outgoing degree 2, so this will lead in $G_{n+1}(X)$ to four vertices, but the edging between them is not entirely determined by $G_n(X)$:

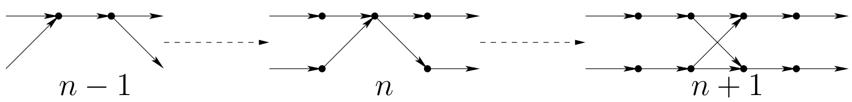
Death Only two edges connect those vertices, no special factor remain after the burst. Such a bispecial factor is called *weak bispecial factor*.



Through Three edges connect those vertices, the left and right special factors do survive. Such a bispecial factor is called *ordinary bispecial factor*.



Mitose Four edges connect those vertices, the left and right special factors are duplicated. Such a bispecial factor is called *strong bispecial factor*.



Hence, strong bispecial factors correspond to forks of $T(X)$ and weak bispecial factors correspond to leaves of $T(X)$. Ordinary bispecial factors and nonbispecial left special factors are not distinguished by $T(X)$, since they both correspond to non-branching nodes.

We have to keep in mind that the "dynamics" we just described between the Rauzy graphs do not correspond to the evolution along the time in (X, S), but has something more linked to the space, letting n increase corresponds much more to a continuous zoom into the structure of (X, S), it is an approximation of (X, S) by subshifts of finite type.

For subshifts whose complexity grows slowly, we saw the importance of left special factors, since they form deconnecting sets and are linked to the complexity.

So, take a minimal subshift of linear complexity such that for some n, $LS_n(X)$ has cardinal K, and suppose that one of them is a strong bispecial factor. Then $LS_{n+1}(X)$ will have cardinal $K+1$. Imagine that one of those left special factors is weak bispecial, then $LS_{n+2}(X)$ will have cardinal K, so if such a phenomenon appears infinitely often, Theorem 1 will allow us to conclude that the subshift has at most K ergodic invariant measures.

Then, imagine the swapped situation, where the weak bispecial factor appears at the scale n and the strong bispecial factor appears at the scale $n+1$: Theorem 1 will give us at most $\mathrm{card}(LS_{n+1}(X)) = K - 1$ ergodic invariant measures.

Wiewed from the scale $n+2$, those two situations are quite the same, and there shouldn't be such a big difference as the number of ergodic invariant measures in the resulting subshifts. Hence, the problem is that, although the phenomena do all appear through Rauzy graphs, they do not necessarily appear with a good synchronization.
In order to avoid such a problem of small missynchronisation, we should group events that appear at "comparable" scales i.e. scales between some n and $n + o(n)$. Such a grouping of events will be done in the next section thanks to the notion of cuts on $T(X)$. Hence, the n should be considered as arbitrary microscales one has to group in order to form scales, the gap between two consecutive scales will depend on the dynamical system.

4 A condition on the tree of left special factors

4.1 A blurred version of theorem 1

Let us begin by a small improvement of theorem 1, that will allow us not to care about local phenomenons and then to consider only as a detail the small missynchronisation described before.
The set of vertices of $G_n(x)$ becomes a metric space as follows:

$$d_n(v_1, v_2) \stackrel{\text{def}}{=} n - \max\{l(w) \mid \#(w, v_1) \geq 1 \text{ and } \#(w, v_2) \geq 1\}$$

Note that if u is a finite word, then $|\#(u, v_1) - \#(u, v_2)| \leq d_n(v_1, v_2)$.
Another classical distance between two vertices of an oriented graph is the length (number of vertices) of the shortest *not necessarily oriented* path between them. The distance we are considering is smaller than that distance since if $v_1 \to v_2$ in $G_n(X)$, then $d(v_1, v_2) \leq 1$.

If $K \geq 1$, a minimal subshift (X, S) is said to be *o-K-deconnectable* if there exists a positive sequence (ρ_n) in $o(n)$, an extraction $\alpha \in\uparrow(\mathbb{N}, \mathbb{N})$ and a constant $K' \geq 1$ such that for all $n \geq 1$ there exists a subset $D_{\alpha(n)} \subset L_{\alpha(n)}(X)$ of at most K vertices such that every path in $G_{\alpha(n)}(X) \setminus V(D_{\alpha(n)}, \rho_{\alpha(n)})$ is of length less than $K'\alpha(n)$. Note that $V(D, \varepsilon) \stackrel{\text{def}}{=} \{v \in L_n(X) \mid (\exists d \in D)(d(d, v) \leq \varepsilon)\}$ denotes the tubular neighboorhood of radius ε around $D \subset L_n(X)$.

Proposition 4. *A o-K-deconnectable minimal subshift has at most K S-invariant ergodic measures.*

This means that we can replace the K deconnecting vertices of theorem 1 by K small balls around them to disconnect the Rauzy graphs.

Proof: It is just a hack in the proof of theorem 1.
The construction of the K possible measures in step 1 and 2 is the same. We can notice that any sequence $d'_{i,\alpha(n)} \in V(\{d'_{i,\alpha(n)}\}, \rho_{\alpha(n)})$ will lead to the same limit μ_i.
We need to be more precise in the step 3: we have the *uniform* bound

$$|\#(u, d_{i,\alpha(n)}) - \#(u, d)| \leq \rho_{\alpha(n)}$$

for any $i \leq K$, $n \in \mathbb{N}$, $u \in L(X)$ and $d \in V(\{d_{i,\alpha(n)}\}, \rho_{\alpha(n)})$.
The result in the upper bounds follows since $\rho_{\alpha(n)}/\alpha(n) \xrightarrow[n\to\infty]{} 0$.

$\square$

Note that a similar result holds if we consider the Hamming $\overline{d}$-distance instead of d, but we don't need it in this paper (in fact the bound we used in the proof won't be uniform anymore, but will depend on the length of the finite word u, so we have to be a little bit more careful).

4.2 A condition on the tree of left special factors

A *cut* is a continuous fonction from $[0, 1]$ to $\mathbb{R}_+$.
The *height* of a cut f is $h(f) \stackrel{\text{def}}{=} min(f)$ and its *oscillation* is $osc(f) \stackrel{\text{def}}{=} max(f) - min(f)$.

Theorem 3. *Let (X, S) be a minimal subshift of linear complexity such that there exists a positive integer K and a sequence (f_n) of cuts such that*

- $h(f_n) \xrightarrow[n\to\infty]{} \infty$
- $osc(f_n) \in o(h(f_n))$
- *a representation of $T(X)$ meets the graph of f_n at at most K points ($n \in \mathbb{N}$).*

Then (X, S) admits at most K ergodic invariant measures.

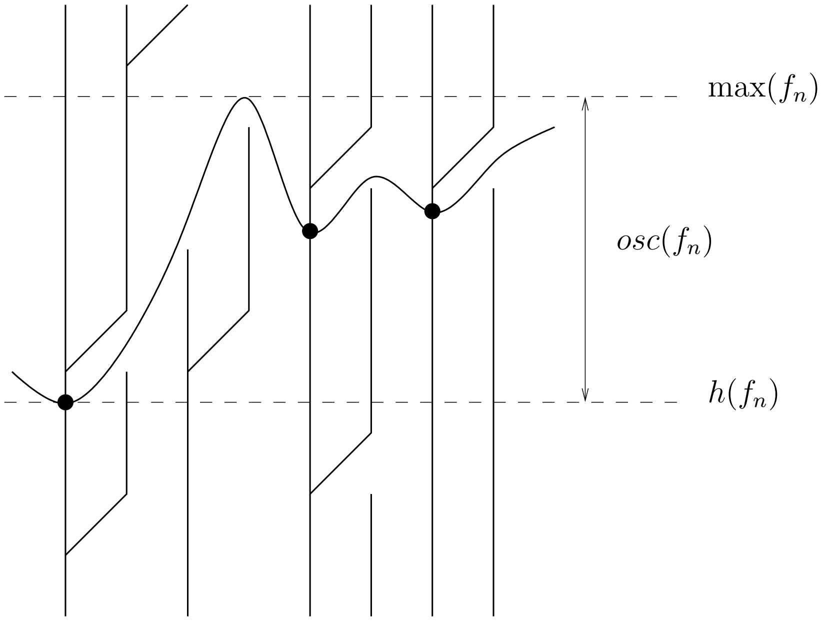


Figure 2: A cut of the tree of the left special factors.

Proof: We will show that (X, S) is o-K-deconnectable.
Up to consider only some of the f_n's, we can assume that the sequence $\alpha(n) \overset{\text{def}}{=} \lceil max(f_n) \rceil$ is an extraction.
Since (X, S) has linear complexity, there exists K' such that for any positive integer n, $p_n(X) \leq K'n$, hence $G_n(X) \setminus LS_n(X)$ does not contain any path of length greater than $K'n$.
Let n be a positive integer.
We group the elements of $LS_{\alpha(n)}(X)$ as follows: to each $v \in LS_{\alpha(n)}(X)$ we choose a point $p(v)$ of $T(X)$ that intersects the graph of f_n. Then we set $v_1 \simeq v_2$ if and only if $p(v_1) = p(v_2)$ (i.e. there are not separated by the cut).
If $v_1 \simeq v_2$, they have a common prefix of length at least $min(f_n)$, hence $d(v_1, v_2) \leq osc(f_n) \leq o(\alpha(n))$.
By construction, $\text{card}(LS_{\alpha(n)}/\simeq) \leq \text{card}(Graph(f_n) \cap T(X)) \leq K$.
A choice of a representative in any class is a $o(\alpha(n))$-deconnecting set.
Therefore (X, S) is o-K-deconnectable and admits at most K ergodic invariant measures.

□

Note that Boshernitzan's result corresponds to the case where the cuts are constant functions with integer values, since the intersection of $T(X)$ with such a cut has cardinal $LS_{f(0)}(X)$. For example, if the situation of Figure 2 appears infinitely often, Theorem 3 will give 3 invariant ergodic measures whereas Boshernitzan's theorem will give 6 invariant ergodic measures.

Conclusion

So, since Theorem 1 depends too much on microscales, Theorem 3 is an attempt to mix different Rauzy graphs in order to synchronize some events that appear at comparable scales.

We can see a minimal subshift as the projective limit of its Rauzy graphs (see figure page). Such a profinite representation was used by Almeida [Alm] to study algebraic invariants associated to minimal subshifts. With that presentation, Rauzy graphs are horizontal slices whereras the tree of left special factors can be understood as its vertebral column (at least for subshifts of small complexities), so those two objects are in some sense transversal to eachother. Hence, the results presented here can be reformulated in that space and we can hope for a much general version of those that will depend on the geometry of that whole space.

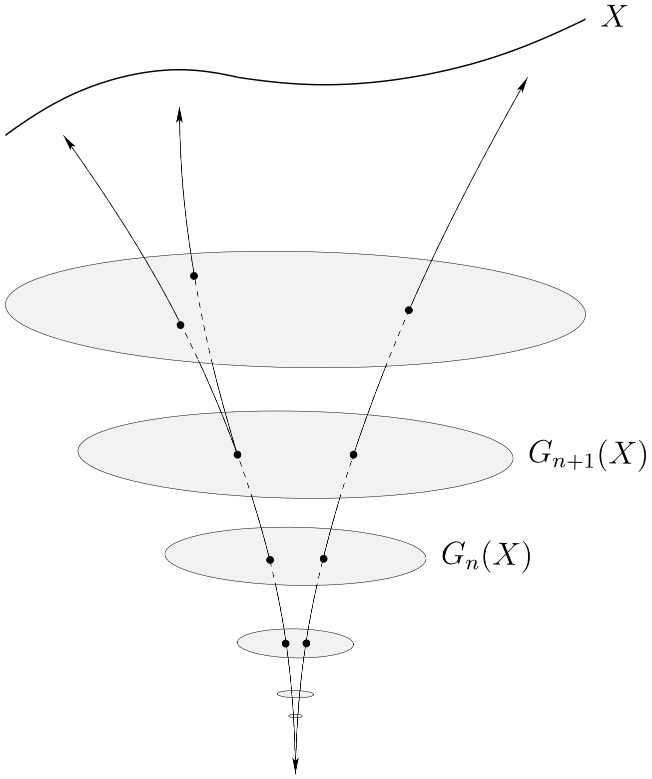


Figure 3: A minimal subshift viewed as a profinite limit of its Rauzy graphs.

For example can we prove in such a symbolic way the result of Katok [Kat] and Veech [Vee1] which asserts that a minimal interval exchange transformation on K intervals admits at most $K/2$ ergodic invariant measures (Boshernitzan's theorem gives a bound equal to $K-1$)?

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