An Exact Reduction Method For Graph Cut Optimization

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Abstract

In few years, graph cuts have now become a fundamental combinatorial optimization tool beside computer vision and graphics communities for solving a wide spectrum of problems. Nevertheless, solving problems with a large number of variables remains computationally expensive and require a high memory storage since underlying graphs involve billion of nodes and even more edges. Except some exact methods [1], [2], [3], the heuristics generally fail to accurately preserve thin structures [4], [5], [6], [7], [8]. In this paper, we propose a test similar to [9] for reducing these graphs. As previous band-based strategies [9], [6], [3], [5], [4], the nodes are typically located in a narrow band surrounding the object edges. In addition, we prove with this test that any node in the non-reduced graph can be safely removed without modifying the maximum flow value, keeping in this way the optimality on the solution. Additionally, we present numerical experiments for segmenting large images which globally depict similar memory gains and segmentation accuracy than those presented in [9].

Index Terms

discrete optimization, graph cuts, segmentation, denoising.

I. Introduction

The introduction of efficient combinatorial optimization tools based on minimum cuts (min-cut) / maximum flow (max-flow) have deeply modified the landscape of computer vision. Indeed, a wide spectrum of ill-posed problems such as segmentation, restoration or dense field estimation consist of a large number of variables and can now be solved with a moderate empirical complexity. Graph cuts have increased the quality and the quantity of low-level analysis tools.

Although graph cuts stay behind the scene during one decade, they become more attractive thanks to a fast max-flow algorithm [10] and efficient heuristics for multi-labels problems [11].

In parallel, technological advances have both exploded the amount and the diversity of data to process. Processing and analyzing these data amounts to solve large scale optimization problems. Despite a low running time and good convergence properties, graph cuts generally fail to solve such problems due to the memory requirements. This problem has been recently adressed by some authors giving rise both to heuristics [7], [4], [5], [6] and exact methods [3], [1], [2].

To our best knowledge, this problem seems to be first tackled in [7]. The strategy adopted is to compute a graph cut in a graph built from a pre-segmentation. While this approach greatly increase the performance of standard graph cuts, the results depend on the algorithm used for computing the pre-segmentation and better results are obtained when over-segmentation occurs.

Band-based methods have been also proposed [4], [5], [6]. A low-resolution of the image is first segmented. Then, the solution is propagated to the finer level by only building the graph in a narrow band surrounding the interpolated foreground/background interface at that resolution. Although this strategy clearly improve the performance of standard graph cuts, it is less accurate to segment thin structures like blood vessels or filaments. Notice that this problem is notably reduced in [5] but still present for low-contrasted details. In [6], smaller graphs are obtained by associating an uncertainty measure to each pixel.

Exact methods have been also investigated [3], [1], [2]. In [3], binary energy functions are minimized for the shape fitting problem with graph cuts in a narrow band while ensuring the optimality on the solution. One makes a band evolve around the object to delineate by expanding it when the min-cut touches its boundary. This process is iterated until the band no longer evolves. Although the algorithm generally converges in few iterations, an initialization is still required.

In [1], a parallel max-flow algorithm yielding a near-linear speedup with the number of processors is presented. While this method achieves good performance on large scale problems, the algorithm is relatively sensitive to the available amount of physical memory and remains less efficient on small graphs.

In [2], the problem is decomposed into optimizable sub-problems, solved independently and updated according to the results of the adjacent problems [2]. This process is iterated until convergence and optimality is guaranteed by Lagrangian decomposition.

Another band-based method was proposed for reducing graphs in binary image segmentation [9]. The graph is progressively built by only adding nodes which locally satisfy a condition. In the manner of previous band-based methods, the graph nodes are typically located in a narrow band surrounding the object edges to segment. This method is able to segment large volumes when standard graph cuts fail while keeping low pixel error. The time for reducing the graph is even sometimes compensated by the time for computing the min-cut in the reduced graph.

In this paper, we pursue the work of [9] and propose a similar test to reduce these graphs by discarding a large amount of nodes during the graph construction. While the cost for evaluating this test is slightly higher compared to [9], we prove that any node satisfying it can be safely removed without modifying the max-flow, keeping in this way the optimality on the solution.

The rest of this document is organized as follows. We first define some notations about flows and cuts in Section II and present the test for reducing the graphs as well as the main theorem of this paper in Section III. The proof of this theorem is detailed in the next sections. This work is completed with experiments for segmenting large grayscale and color images in Section VII.

II. NOTATIONS AND PRELIMINARIES

We consider a set of pixels $\mathcal{P} \subset \mathbb{Z}^d$, for a positive integer d. We consider two terminal nodes s and t and the set of nodes

$$\mathcal{V} \stackrel{\text{def}}{=} \mathcal{P} \cup \{s, t\}.$$

We consider a set of directed edges $\mathcal{E} \subset (\mathcal{V} \times \mathcal{V})$ such that $(\mathcal{V}, \mathcal{E})$ is a simple directed graph. We also assume that for every $p \in \mathcal{P}$,

$$(p, s) \notin \mathcal{E} \text{ and } (t, p) \notin \mathcal{E}. \tag{1}$$

We denote the neighbors of any nodes $p \in \mathcal{V}$ by

$$\sigma_{\mathcal{E}}(p) \stackrel{\text{def}}{=} \{q \in \mathcal{V}, (p, q) \in \mathcal{E} \text{ or } (q, p) \in \mathcal{E}\}.$$

We denote a walk of positive length $l \in \mathbb{N}$ by $p_0 - p_1 - \ldots - p_l$, where $p_i \in \mathcal{V}$, for all $i \in \{0, \ldots, l\}$, and $(p_i, p_{i+1}) \in \mathcal{E}$, for all $i \in \{0, \ldots, l-1\}$. We also remind that a closed walk is such that $p_0 = p_l$. We denote by $W_a(p, q)$ the set containing all the walks starting at $p \in \mathcal{V}$ and ending at $q \in \mathcal{V}$.

We define the capacities as a mapping $c : (\mathcal{V} \times \mathcal{V}) \to \mathbb{R}^+$ and denote the capacity of any edge $(p, q) \in (\mathcal{V} \times \mathcal{V})$ by

$$c_{p,q} \geq 0.$$

Although c is defined for any $(p, q) \in (\mathcal{V} \times \mathcal{V})$, we always set

$$c_{p,q} = 0, \text{ when } (p, q) \notin \mathcal{E}, \tag{2}$$

so-that non-null capacities are only defined on existing edges. The purpose for extending the definition capacities to elements of $(\mathcal{V} \times \mathcal{V}) \setminus \mathcal{E}$ in this manner is to simplify notations in many upcoming summations and equations.

We assume, without loss of generality (see [12]), that capacities are such that for every $p \in \mathcal{P}$

$$c_{s,p} \neq 0 \quad \Rightarrow \quad c_{p,t} = 0. \tag{3}$$

We therefore summarize the capacities of the edges linked to the terminal nodes and set for all $p \in \mathcal{P}$

$$c_p = c_{s,p} - c_{p,t}. \tag{4}$$

For any $S \subset \mathcal{P}$, we denote the value of the s-t cut $(S \cup \{s\}, (\mathcal{P} \setminus S) \cup \{t\})$ in $\mathcal{G}$ by $\mathrm{val}_{\mathcal{G}}(S)$. We remind that

$$\mathrm{val}_{\mathcal{G}}(S) = \sum_{\substack{p \in S \cup \{s\} \\ q \notin S \cup \{s\}}} c_{p,q}.$$

Notice that, we have not clarified that $(p,q) \in \mathcal{E}$ in the above summation thanks to (2).

We also define flows as any mapping $f : (\mathcal{V} \times \mathcal{V}) \to \mathbb{R}^+$ satisfying the capacity constraints

$$0 \le f_{p,q} \le c_{p,q}, \text{ for all } (p,q) \in (\mathcal{V} \times \mathcal{V}), \tag{5}$$

and the flow conservation

$$\sum_{q \in \sigma_{\mathcal{E}}(p)} f_{q,p} = \sum_{q \in \sigma_{\mathcal{E}}(p)} f_{p,q}. \tag{6}$$

Again, (2) and (5) guarantee that

$$f_{p,q} = 0, \text{ for any } (p,q) \notin \mathcal{E}. \tag{7}$$

This is the reason why we do not clarify that $(q,p) \in \mathcal{E}$ (resp. $(p,q) \in \mathcal{E}$) in the left (resp. right) hand side sum in (6). As usual, the value of the flow f in $\mathcal{G}$ is defined by

$$\mathrm{val}_{\mathcal{G}}(f) = \sum_{p \in \sigma_{\mathcal{E}}(s)} f_{s,p}. \tag{8}$$

Notice that we use the same notation for the value of a flow and the value of a s-t cut in $\mathcal{G}$. This abuse of notation will never be ambiguous once in context. As for capacities, we summarize the flow passing through the edges linked to the terminal nodes and set for all $p \in \mathcal{P}$

$$f_p = f_{s,p} - f_{p,t}. \tag{9}$$

As is well known (and can easily be shown by induction on the cardinality of S), for any flow f and any $S \subset \mathcal{V}$ the flow entering S is equal to the flow exiting S:

$$\sum_{\substack{p \in S \\ q \notin S}} f_{q,p} = \sum_{\substack{p \in S \\ q \notin S}} f_{p,q}. \tag{10}$$

Considering (3), (5) and (9), we can rewrite (10) and obtain that

$$\text{for any } S \subset \mathcal{P}, \qquad \sum_{p\in S} f_p + \sum_{\substack{p\in S\\ q\notin S}} (f_{q,p} - f_{p,q}) = 0, \tag{11}$$

We call max-flow any solution f^* of the linear program

$$\begin{cases} \max_f \quad \mathrm{val}_{\mathcal{G}}(f), \\ \text{under the constraints (5) and (6).} \end{cases}$$

As shown in [13], the value of the max-flow is equal to the value of the min-cut:

$$\mathrm{val}_{\mathcal{G}}(f^*) = \min_{S\subset\mathcal{P}} \quad \mathrm{val}_{\mathcal{G}}(S).$$

Remind that when minimizing a pairwise Markov Random Field of the form

$$E(u) = \beta \sum_{p\in\mathcal{P}} E_p(u_p) + \sum_{(p,q)\in(\mathcal{P}\times\mathcal{P})} E_{p,q}(u_p, u_q), \qquad \beta \in \mathbb{R}^+, \tag{12}$$

among $u \in \{0,1\}^{\mathcal{P}}$, a graph $\mathcal{G}$ is built such that for any $S \subset \mathcal{P}$, we have

$$\mathrm{val}_{\mathcal{G}}(S) = E(u^S) + K, \tag{13}$$

for some additional constant $K \in \mathbb{R}$ and where $u^S \in \{0,1\}^{\mathcal{P}}$ is defined by

$$u_p^S = \begin{cases} 0 & \text{if} \quad p \notin S \\ 1 & \text{if} \quad p \in S \end{cases}, \qquad \forall p \in \mathcal{P}.$$

The min-cut in $\mathcal{G}$ corresponds to a minimizer of (12) and can be efficiently computed by using a max-flow algorithm such as [10]. Additionally, when the terms $E_{p,q}(.)$ are submodular, [12] describe a construction of $\mathcal{G}$ satisfying (13) and prove that (12) can be globally minimized.

Throughout the paper, we consider a fixed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, c)$, with $\mathcal{V}$, $\mathcal{E}$ and c as above. Our purpose is to exhibit a maximum flow satisfying some condition for this graph. Along this paper, we also denote $B \subset \mathbb{Z}^d$ and assume that B and $\mathcal{G}$ are such that

$$\forall p \in \mathcal{P}, \quad (\sigma_{\mathcal{E}}(p) \cap \mathcal{P}) \ \subset \ B_p, \tag{14}$$

where

$$B_p \stackrel{\text{def}}{=} \{p + q, q \in B\}. \tag{15}$$

In practice, we typically think of B as a ball centered at the origin. In such a case, (14) means that neighbors in the graph $\mathcal{G}$ are close to each other in $\mathbb{Z}^d$.

III. Reduction test

Theorem 1 *Let $\mathcal{G}$ be the graph defined in Section II, let B satisfy* (14) *and let us assume that $p \in \mathcal{P}$ satisfies*

$$\begin{cases} \textit{either} \quad \forall q \in B_p, \quad c_q \geq 0 \quad \textit{and} \quad c_q \geq \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \notin B_p}} c_{q,q'}, \\ \textit{or} \quad \forall q \in B_p, \quad c_q \leq 0 \quad \textit{and} \quad c_q \leq \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \notin B_p}} c_{q,q'}. \end{cases} \tag{16}$$

Then, there exists a max-flow f in $\mathcal{G}$ such that

$$\forall q \in \sigma_{\mathcal{E}}(p), \qquad f_{p,q} = f_{q,p} = 0.$$

As a consequence, removing the node p from the graph $\mathcal{G}$ does not modify its max-flow value.

The proof of this theorem is contained in Section IV, Section V and Section VI. For simplicity, we only prove the theorem when the node p satisfies the first condition of (16).

Algorithmically, the above theorem guarantees that we can test every node, during the graph construction, before it is added to the graph. If the node satisfies (16), it is not useful to the max-flow evaluation and can be removed without alteration of the max-flow value.

IV. Avoiding useless flow on closed walks

In this section we remind a known result. We also prove it so-that the paper is self contained.

Proposition 1 *Let $\mathcal{G}$ be the graph defined in Section II. There exists a max-flow f in $\mathcal{G}$ satisfying*

$$\begin{cases} \textit{for any length } l \textit{ and any closed walk } p_0 - p_1 - \ldots - p_l \textit{ of length } l \textit{ in } \mathcal{G}, \\ \textit{there exists } i \in \{0, \ldots, l\} \textit{ such that } f_{p_i,p_{i+1}} \leq f_{p_{i+1},p_i} \textit{ where we denote } p_{l+1} = p_0. \end{cases} \tag{17}$$

Proof. Let f be a max-flow in $\mathcal{G}$. For any l and any closed walk $w = p_0 - p_1 - \ldots - p_l$ of length l in $\mathcal{G}$, we set $p_{l+1} = p_0$ and denote $(P_{f,w})$ the statement:

$$(P_{f,w}) : \forall i \in \{0, \ldots, l\}, f_{p_i,p_{i+1}} > f_{p_{i+1},p_i}.$$

In particular, a closed walk w satisfying the previous statement to take reverse edges. We also denote

$$W(f) \stackrel{\text{def}}{=} \{w, w \text{ is a closed walk satisfying } (P_{f,w})\}.$$

Notice first that if

$$\#W(f) = 0, \tag{18}$$

where $\#$ denotes the cardinality of a set, the flow f necessarily satisfies (17).

We show, in the remaining of the proof, that if f is such that $\#W(f) > 0$, there exist f' such that

$$\#W(f') < \#W(f),$$

where $\#$ denotes the cardinality of a set. Since for any max-flow f the set $W(f)$ is finite, any initial max-flow lead to a max-flow satisfying (18) (and therefore (17)) after a finite number of such recursion.

Let us now assume that f is such that $\#W(f) > 0$. Let us also consider a closed walk $w = p_0 - p_1 - \ldots - p_l \in W(f)$.

We denote $p_{l+1} = p_0$ and

$$\delta \stackrel{\text{def}}{=} \min_{i \in \{0,\ldots,l\}} (f_{p_i,p_{i+1}} - f_{p_{i+1},p_i}).$$

Since w satisfies $(P_{f,w})$, we have $\delta > 0$.

We define the mapping $f' : (\mathcal{V} \times \mathcal{V}) \to \mathbb{R}^+$ such that for all $(p, q) \in (\mathcal{V} \times \mathcal{V})$:

$$f'_{p,q} = \begin{cases} f_{p,q} - f_{q,p} - \delta & \text{, if } (p,q) = (p_i, p_{i+1}), \text{ for some } 0 \le i \le l \\ 0 & \text{, if } (p,q) = (p_{i+1}, p_i), \text{ for some } 0 \le i \le l \\ f_{p,q} & \text{, otherwise.} \end{cases} \tag{19}$$

Notice that this definition is not ambiguous. Indeed, we cannot simultaneously have $(p, q) = (p_i, p_{i+1})$ and $(p, q) = (p_{j+1}, p_j)$ for some $i \neq j$ since w satisfies $(P_{f,w})$.

Also, since f is a flow in $\mathcal{G}$, we clearly have for all $(p, q) \in (\mathcal{V} \times \mathcal{V})$

$$0 \le f'_{p,q} \le c_{p,q}.$$

In order to prove the flow conservation, we consider $p \in \mathcal{V}$. Let us first assume that $p \neq p_i$, for all $i \in \{0, \ldots, l\}$. Then (19) guarantees that $f'_{p,q} = f_{p,q}$ for all $q \in \sigma_{\mathcal{E}}(p)$ and we trivially get

$$\sum_{q \in \sigma_{\mathcal{E}}(p)} f'_{q,p} = \sum_{q \in \sigma_{\mathcal{E}}(p)} f'_{p,q}.$$

Let us now assume that there exists $i \in \{0, \ldots, l\}$ such that $p = p_i$. We denote

$$I = \{j \in \{0, \ldots, l\}, p = p_j\}$$

and $p_{-1} = l$. We have

$$\sum_{q \in \sigma_{\mathcal{E}}(p)} (f'_{q,p} - f'_{p,q}) = \sum_{\substack{q \in \sigma_{\mathcal{E}}(p) \\ q \neq p_{j+1}, \forall j \in I \\ q \neq p_{j-1}, \forall j \in I}} (f'_{q,p} - f'_{p,q}) + \sum_{j \in I} (f'_{p_{j+1},p_j} - f'_{p_j,p_{j+1}}) + \sum_{j \in I} (f'_{p_{j-1},p_j} - f'_{p_j,p_{j-1}})$$

Using (19), we obtain for each term

$$\sum_{\substack{q \in \sigma_{\mathcal{E}}(p) \\ q \neq p_{j+1}, \forall j \in I \\ q \neq p_{j-1}, \forall j \in I}} (f'_{q,p} - f'_{p,q}) = \sum_{\substack{q \in \sigma_{\mathcal{E}}(p) \\ q \neq p_{j+1}, \forall j \in I \\ q \neq p_{j-1}, \forall j \in I}} (f_{q,p} - f_{p,q}), \tag{20}$$

$$\sum_{j \in I} (f'_{p_{j+1},p_j} - f'_{p_j,p_{j+1}}) = \sum_{j \in I} -(f_{p_j,p_{j+1}} - f_{p_{j+1},p_j} - \delta), \tag{21}$$

and

$$\sum_{j \in I} (f'_{p_{j-1},p_j} - f'_{p_j,p_{j-1}}) = \sum_{j \in I} (f_{p_{j-1},p_j} - f_{p_j,p_{j-1}} - \delta). \tag{22}$$

Summing (20), (21), (22) and simplifying, we finally get

$$\begin{aligned}
\sum_{q \in \sigma_{\mathcal{E}}(p)} (f'_{q,p} - f'_{p,q}) &= \sum_{\substack{q \in \sigma_{\mathcal{E}}(p) \\ q \neq p_{j+1}, \forall j \in I \\ q \neq p_{j-1}, \forall j \in I}} (f_{q,p} - f_{p,q}) + \sum_{j \in I} \big((f_{p_{j+1},p_j} - f_{p_j,p_{j+1}}) + (f_{p_{j-1},p_j} - f_{p_j,p_{j-1}}) \big) \\
&= \sum_{q \in \sigma_{\mathcal{E}}(p)} (f_{q,p} - f_{p,q}) \\
&= 0
\end{aligned}$$

As a conclusion, f' is a flow. It is of course a max-flow. Indeed, (1) and (7) guarantee that $f_{p,s} = 0$, for all $p \in \mathcal{P}$. Since w satisfies $(P_{f,w})$, this ensures that

$$s \neq p_i, \forall i \in \{0, \dots, l\}.$$

Using (8) and (19), we finally get

$$\mathrm{val}_{\mathcal{G}}(f') = \mathrm{val}_{\mathcal{G}}(f).$$

We still need to show that

$$\#W(f') < \#W(f).$$

With that in mind, we consider $w' = p'_0 - p'_1 - \ldots - p'_{l'} \in W(f')$. Denoting $p'_{l'+1} \stackrel{\text{def}}{=} p'_0$, we know that for any $j \in \{0, \dots, l'\}$

$$0 < (f'_{p'_j,p'_{j+1}} - f'_{p'_{j+1},p'_j}).$$

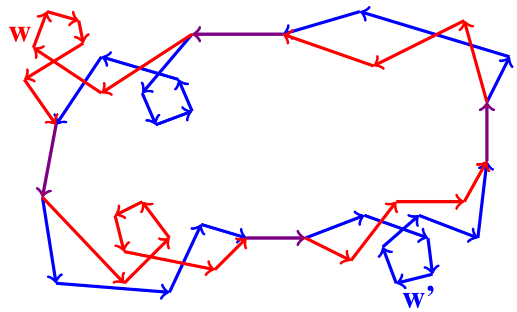

Fig. 1: A situation where the closed walks $w \in W(f)$ (red) and $w' \in W(f')$ (blue) in $\mathcal{G}$ both satisfy (19). The edges in purple correspond to the intersection of w and w'. Observe that w cannot follows reverse edges in w' and conversely since (19) must be satisfied.
figure

Together with (19), this guarantees that for any $j \in \{0, \ldots, l'\}$

$$(p'_j, p'_{j+1}) \neq (p_{i+1}, p_i), \text{ for all } i \in \{0, \ldots, l\}.$$

Such a situation is illustrated in Figure 1. Using (19) again, we therefore necessarily have

$$0 < (f'_{p'_j, p'_{j+1}} - f'_{p'_{j+1}, p'_j}) \leq (f_{p'_j, p'_{j+1}} - f_{p'_{j+1}, p'_j}).$$

This means that $w' \in W(f)$ and, as a result,

$$W(f') \subset W(f).$$

In order to show that this inclusion is strict, we denote

$$i_0 \in \underset{i \in \{0, \ldots l\}}{\text{argmin}} \ (f_{p_i, p_{i+1}} - f_{p_{i+1}, p_i}).$$

Using (19), we trivially obtain that

$$f'_{p_{i_0}, p_{i_0+1}} = f'_{p_{i_0+1}, p_{i_0}} = 0,$$

and therefore $w \notin W(f')$. This concludes the proof. $\square$

V. Avoiding useless traversing flow

Throughout this section, we consider a graph $\mathcal{G}$ as constructed in Section II and a max-flow f in $\mathcal{G}$ satisfying (17). We also consider $p \in \mathcal{P}$ such that

$$\forall q \in B_p, f_q \geq 0,$$

where B_p is defined in (15). [1]

The purpose of this section is to establish a sufficient condition so-that f can be modified in such a way that

$$f_{p,q} \geq f_{q,p}, \text{ for all } q \in \sigma_{\mathcal{E}}(p).$$

In words, the node p globally sends more flow to its neighbors than it can receive from them.

In order to do so, we consider

$$\Sigma_i(p) = \{q \in \mathcal{P}, \exists p_0 - \ldots - p_l \in W_a\left(q,p\right) \text{ such that } \forall i \in \{0,\ldots,l-1\}, f_{p_i,p_{i+1}} > f_{p_{i+1},p_i}\},$$

and

$$\Sigma_o(p) = \{q \in B_p, \exists p_0 - \ldots - p_l \in W_a\left(p,q\right) \text{ such that } \forall i \in \{0,\ldots,l-1\}, f_{p_i,p_{i+1}} > f_{p_{i+1},p_i}\},$$

where we remind that $W_a\left(q,p\right)$ (resp. $W_a\left(p,q\right)$) contains all the walks starting at q (resp. p) and ending at p (resp. q). Let us first notice that, since f satisfies (17),

$$\forall q \in \left(\Sigma_i(p) \cap \sigma_{\mathcal{E}}(p)\right), \quad f_{q,p} \geq f_{p,q} \tag{23}$$

and

$$\forall q \in \left(\Sigma_o(p) \cap \sigma_{\mathcal{E}}(p)\right), \quad f_{p,q} \geq f_{q,p}. \tag{24}$$

Similarly, since f satisfies (17), we have

$$\Sigma_i(p) \cap \Sigma_o(p) = \emptyset. \tag{25}$$

Moreover,

$$p \notin \Sigma_i(p) \text{ and } p \notin \Sigma_o(p).$$

For simplicity, we denote

$$\Sigma^- = \Sigma_i(p) \text{ and } \Sigma^+ = \Sigma_o(p) \cup \{p\}.$$

[1] Notice that all the content of this section could be adapted to a situation where $f_q \leq 0$, for all $q \in B_p$.

Also, since f satisfies (17), we have

$$\forall q \in \Sigma^-, \forall q' \in \Sigma^+, \qquad f_{q,q'} \geq f_{q',q}. \tag{26}$$

Otherwise, we could easily build a closed walk contradicting (17). We also denote

$$\mathcal{P}' = \Sigma^- \cup \Sigma^+ \quad , \quad \mathcal{V}' = \mathcal{P}' \cup \{s, t\} \tag{27}$$

and construct the graph

$$\mathcal{G}' = (\mathcal{V}', \mathcal{E}', c')\,,$$

where $\mathcal{E}'$ and c' are defined below. We set

$$\mathcal{E}' = \mathcal{E}'_t \cup (\mathcal{E}'_n \cap \mathcal{E}^T), \tag{28}$$

where $\mathcal{E}^T = \{(q, q'), (q', q) \in \mathcal{E}\}$ and with

$$\mathcal{E}'_t = \{(q,t), \text{ with } q \in \Sigma^- \text{ such that } f_q \geq 0\} \;\; \bigcup \;\; \big(\{s\} \times \Sigma^+\big) \tag{29}$$

and

$$\mathcal{E}'_n = \big(\Sigma^+ \times \Sigma^+\big) \;\; \bigcup \;\; \big((\Sigma^- \cup \{p\}) \times (\Sigma^- \cup \{p\})\big)\,. \tag{30}$$

The capacities c' are defined by

$$c'_{q,t} = f_q \qquad , \text{ for } q \in \Sigma^- \text{ such that } f_q \geq 0, \tag{31}$$

$$c'_{s,q} = c_q - f_q \qquad , \text{ for } q \in \Sigma^+, \tag{32}$$

and

$$c'_{q,q'} = \begin{cases} f_{q',q} - f_{q,q'} & , \text{ if } f_{q',q} > f_{q,q'} \\ 0 & , \text{ otherwise} \end{cases} \qquad , \text{ for } (q,q') \in (\mathcal{E}'_n \cap \mathcal{E}^T). \tag{33}$$

Notice that there exist some nodes in Σ^- which are linked to no terminals. An example of configuration with B_p and the graph $\mathcal{G}'$ is outlined in Figure 2. As in Section II, we artificially extend all the capacities c' and set

$$c'_{q,q'} = 0, \text{ for all } (q,q') \in ((\mathcal{V}' \times \mathcal{V}') \setminus \mathcal{E}')\,.$$

Notice that, in the graph $\mathcal{G}'$ all the flow sent by s goes in Σ^+ and all the flow arriving at t comes from Σ^-. Moreover, all the edges between Σ^+ and Σ^- contain p.

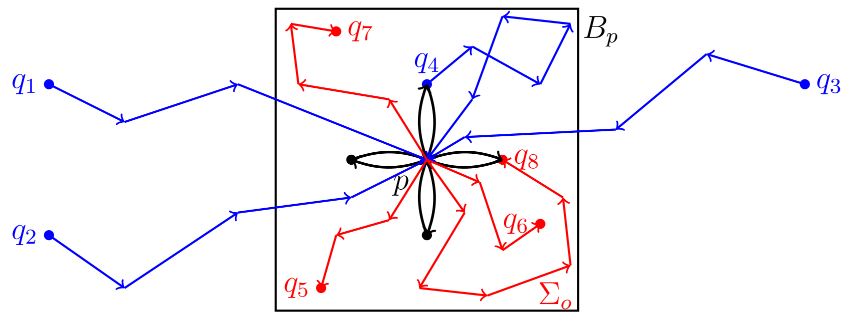

Fig. 2: An example of graph $\mathcal{G}'$ where the flow sent by s goes in the nodes $q_1, q_2, q_3, q_4 \in \Sigma^-$ (blue) and all the flow arriving at t comes from the nodes $q_5, q_6, q_7, q_8 \in \Sigma^+$ (red). Notice that the nodes of Σ^+ are bounded to B_p whereas the nodes of Σ^- are a subset of $\mathcal{P}$.
figure

Also, for any $S \subset \mathcal{P}'$, we denote the value of the s-t cut $(S \cup \{s\}, (\mathcal{P}' \setminus S) \cup \{t\})$ in $\mathcal{G}'$ by

$$\mathrm{val}_{\mathcal{G}'}(S) = \sum_{\substack{q \in (S \cup \{s\}) \\ q' \notin (S \cup \{s\})}} c'_{q,q'}.$$

Using (31), (32) and (33), we find

$$\mathrm{val}_{\mathcal{G}'}(S) = E_1 + E_2 + E_3,$$

where we write

$$E_1 = \sum_{q \in (\Sigma^+ \setminus S)} c'_{s,q} \quad , \quad E_2 = \sum_{q \in (\Sigma^- \cap S)} c'_{q,t} \quad \text{and} \quad E_3 = \sum_{\substack{q \in S \\ q' \in (\mathcal{P}' \setminus S)}} c'_{q,q'}. \tag{34}$$

In particular, using (25) and (27), we have

$$\mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right) = \sum_{\substack{q \in \Sigma^+ \\ q' \in \Sigma^-}} c'_{q,q'},$$

which, using (30), (25) becomes

$$\mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right) = \sum_{q \in \Sigma^-} c'_{p,q}.$$

Finally, we obtain using (33) and (23)

$$\mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right) = \sum_{q \in \Sigma^-} (f_{q,p} - f_{p,q}). \tag{35}$$

The following proposition holds.

Proposition 2 *Let $\mathcal{G}'$ be the graph constructed in Section V. For any $S \subset \mathcal{P}'$,*

$$\mathrm{val}_{\mathcal{G}'}(S) \geq \mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right) + \sum_{q \in \Sigma^+ \setminus (S \cup \{p\})} \left[c_q + \sum_{q' \notin \Sigma^+} (f_{q',q} - f_{q,q'}) \right]. \quad (36)$$

Proof. Let us first decompose E_3 according to

$$E_3 = E_1' + E_2' + E_3' + E_4',$$

with

$$E_1' = \sum_{\substack{q \in (S \cap \Sigma^+) \\ q' \in (\Sigma^+ \setminus S)}} c'_{q,q'} \qquad , E_2' = \sum_{\substack{q \in (S \cap \Sigma^+) \\ q' \in (\Sigma^- \setminus S)}} c'_{q,q'}$$

$$E_3' = \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^+ \setminus S)}} c'_{q,q'} \qquad , E_4' = \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^- \setminus S)}} c'_{q,q'}$$

We rewrite, using (33),

$$E_1' = \sum_{\substack{q \in (S \cap \Sigma^+) \\ q' \in (\Sigma^+ \setminus S) \\ f_{q',q} > f_{q,q'}}} (f_{q',q} - f_{q,q'}) \qquad , E_2' = \sum_{\substack{q \in (S \cap \Sigma^+) \\ q' \in (\Sigma^- \setminus S) \\ (q,q') \in \mathcal{E}', f_{q',q} > f_{q,q'}}} (f_{q',q} - f_{q,q'}) \quad (37)$$

$$E_3' = \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^+ \setminus S) \\ (q,q') \in \mathcal{E}', f_{q',q} > f_{q,q'}}} c'_{q,q'} \qquad , E_4' = \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^- \setminus S) \\ f_{q',q} > f_{q,q'}}} (f_{q',q} - f_{q,q'}) \quad (38)$$

Using (30) and (25), then (33) and (23), we immediately find that

$$E_2' = \begin{cases} \sum_{q \in (\Sigma^- \setminus S)} (f_{q,p} - f_{p,q}) & \text{, if } p \in S \\ 0 & \text{, otherwise,} \end{cases} \qquad \text{and} \qquad E_3' = 0. \quad (39)$$

Moreover, since the total amount of flow entering and exiting $(S \cap \Sigma^-)$ are equal, we have (see (11))

$$\sum_{\substack{q \in (S \cap \Sigma^-) \\ f_q \geq 0}} f_q + \sum_{\substack{q \in (S \cap \Sigma^-) \\ f_q < 0}} f_q + \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \notin (S \cap \Sigma^-)}} (f_{q',q} - f_{q,q'}) = 0$$

Moreover, if we decompose the last term and reorganize the equation we obtain

$$\sum_{\substack{q \in (S \cap \Sigma^-) \\ f_q \geq 0}} f_q + \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^- \setminus S)}} (f_{q',q} - f_{q,q'}) = - \sum_{\substack{q \in (S \cap \Sigma^-) \\ f_q < 0}} f_q - \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in \Sigma^+}} (f_{q',q} - f_{q,q'})$$

Together with the definition of E_2 in (34), the definition of E'_4 in (38) and (31) this leads to

$$\begin{aligned} E_2 + E'_4 &\geq \sum_{\substack{q \in (S \cap \Sigma^-) \\ f_q \geq 0}} f_q + \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^- \setminus S)}} (f_{q',q} - f_{q,q'}) \\ &\geq -\sum_{\substack{q \in (S \cap \Sigma^-) \\ f_q < 0}} f_q - \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in \Sigma^+}} (f_{q',q} - f_{q,q'}) \\ &\geq \sum_{q \in (S \cap \Sigma^-)} (f_{q,p} - f_{p,q}) + \sum_{\substack{q \in (S \cap \Sigma^-) \\ q' \in (\Sigma^+ \setminus \{p\})}} (f_{q,q'} - f_{q',q}). \end{aligned}$$

Then, using (26), we immediately obtain

$$E_2 + E'_4 \geq \sum_{q \in (S \cap \Sigma^-)} (f_{q,p} - f_{p,q}).$$

Together with (39) and (35), this leads to the following intermediate result:

$$E_2 + E'_2 + E'_3 + E'_4 \geq \begin{cases} \mathrm{val}_{\mathcal{G}'}(\Sigma^+) & \text{, if } p \in S \\ \sum_{q \in (S \cap \Sigma^-)} (f_{q,p} - f_{p,q}) & \text{, otherwise.} \end{cases} \tag{40}$$

In order to finish the proof, let us first notice that using the definition of E_1 in (34), (32) and the definition of E'_1 in (37)

$$E_1 + E'_1 \geq \sum_{q \in (\Sigma^+ \setminus S)} (c_q - f_q) + \sum_{\substack{q \in (S \cap \Sigma^+) \\ q' \in (\Sigma^+ \setminus S)}} (f_{q',q} - f_{q,q'}) \tag{41}$$

Expressing that the total amount of flow entering and exiting $(\Sigma^+ \setminus S)$ are equal, we have (see (11))

$$\sum_{q \in (\Sigma^+ \setminus S)} f_q + \sum_{\substack{q \in (\Sigma^+ \setminus S) \\ q' \in (\Sigma^+ \cap S)}} (f_{q',q} - f_{q,q'}) + \sum_{\substack{q \in (\Sigma^+ \setminus S) \\ q' \notin \Sigma^+}} (f_{q',q} - f_{q,q'}) = 0.$$

Together with (41), this guarantees that

$$\begin{aligned} E_1 + E'_1 &\geq \sum_{q \in (\Sigma^+ \setminus S)} c_q + \sum_{\substack{q \in (\Sigma^+ \setminus S) \\ q' \notin \Sigma^+}} (f_{q',q} - f_{q,q'}), \\ &\geq \sum_{q \in (\Sigma^+ \setminus S)} \left[c_q + \sum_{q' \notin \Sigma^+} (f_{q',q} - f_{q,q'}) \right] \end{aligned} \tag{42}$$

When $p \in S$, by combining the latter result with (40), we immediately get (36). If $p \notin S$, (42) can be rewritten using (35)

$$E_1 + E'_1 \geq \sum_{q \in (\Sigma^+ \setminus (S \cup \{p\}))} \left[c_q + \sum_{q' \notin \Sigma^+} (f_{q',q} - f_{q,q'}) \right] + c_p + \mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right).$$

Since $c_p \geq 0$, and (40) and (23) guarantee that $E_2 + E'_2 + E'_3 + E'_4 \geq 0$, this ensures that (36) holds even when $p \notin S$ and concludes the proof. □

All along the remaining of this Section, we consider a max-flow f' in $\mathcal{G}'$. Notice also that $\mathcal{G}'$ satisfies (1), (3). Therefore, as in Section II, we denote

$$f'_q = f'_{s,q} - f'_{q,t},$$

for all $q \in \mathcal{P}'$. We also artificially extend the flow f' and set

$$f'_{q,q'} = 0, \text{ for all } (q,q') \in ((\mathcal{V}' \times \mathcal{V}') \setminus \mathcal{E}').$$

We are now going to combine f and f' in order to build a mapping $f'' : \mathcal{E} \to \mathbb{R}$ which will turn out to be a max-flow in $\mathcal{G}$ such that

$$f''_{p,q} \geq f''_{q,p} = 0 \quad , \forall q \in \sigma_{\mathcal{E}}(p).$$

Let us begin with the definition of f''. We set

$$f''_{q,q'} = f_{q,q'} \qquad , \text{ for } (q,q') \notin \mathcal{E}', q \neq s, q' \neq t, \tag{43}$$

$$\begin{cases} f''_{s,q} = 0 & \text{and } f''_{q,t} = -f_q \quad , \text{ for } q \in \mathcal{P}' \text{ such that } f_q < 0 \\ f''_{s,q} = f_q + f'_q & \text{and } f''_{q,t} = 0 \quad , \text{ for } q \in \mathcal{P}' \text{ such that } f_q \geq 0 \end{cases} \tag{44}$$

$$f''_{q',q} = \begin{cases} \underbrace{f_{q',q} - f_{q,q'}}_{c'_{q,q'}} - f'_{q,q'} & , \text{ if } f_{q',q} > f_{q,q'}, \\ 0 & , \text{ otherwise} \end{cases} \quad , \text{ for } (q',q) \in \mathcal{P}'^2 \text{ such that } (q,q') \in \mathcal{E}'. \tag{45}$$

Notice that the equations (43), (44) and (45) permit to define $f''_{q,q'}$ for all $(q,q') \in \mathcal{E}$. Once again, we extend f'' outside $\mathcal{E}$ and set

$$f''_{q,q'} = 0, \text{ for all } (q,q') \in ((\mathcal{V} \times \mathcal{V}) \setminus \mathcal{E}).$$

We also denote

$$f''_q = f''_{s,q} - f''_{q,t} \quad , \forall q \in \mathcal{P}.$$

Notice that, since $f'_q = 0$ for all $q \notin \mathcal{P}'$ as well as for $q \in \mathcal{P}'$ such that $f_q < 0$ (see (31) and (32)), we always have, according to (43) and (44),

$$f''_q = f_q + f'_q \quad , \forall q \in \mathcal{P}. \tag{46}$$

Proposition 3 *The mapping* $f'' : (\mathcal{V} \times \mathcal{V}) \to \mathbb{R}$ *is a max-flow in* $\mathcal{G}$.

Proof. Let us first show that f'' satisfies the capacity constraints. Let $(q', q) \in \mathcal{E}$.

- If q or $q' \notin \mathcal{P}'$, $q \neq s$, $q' \neq s$: then $(q', q) \notin \mathcal{E}'$ and using (43) we have

$$0 \leq f''_{q',q} = f_{q',q} \leq c_{q',q}.$$

- If $q' \in \Sigma^-$ and $q = s$ or t:
  - If moreover $f_{q'} < 0$, then using (44), $0 \leq f''_{s,q'} = 0 \leq c_{s,q'}$ and $0 \leq f''_{q',t} = f_{q',t} \leq c_{q',t}$.
  - If $f_{q'} \geq 0$, then using (44) and (32), we find that $0 \leq f''_{s,q'} = f_{s,q'} - f'_{q',t} \leq c_{s,q'}$ and $0 \leq f''_{q',t} = 0 \leq c_{q',t}$.
- If $q' \in \Sigma^+$ and $q = s$ or t: since $q' \in B_p$, we necessarily have $f_{q'} \geq 0$, then using (44) and (32), we have $0 \leq f''_{s,q'} = f_{s,q'} + f'_{s,q'} \leq c_{s,q'}$ and $0 \leq f''_{q',t} = 0 \leq c_{q',t}$.
- If $(q', q) \in (\mathcal{P}' \times \mathcal{P}')$:
  - If moreover $f_{q',q} \leq f_{q,q'}$, then (45) guarantees $0 \leq f''_{q',q} = 0 \leq c_{q',q}$.
  - If $f_{q',q} > f_{q,q'}$, using (33), we have

$$0 \leq f'_{q,q'} \leq c'_{q,q'} = f_{q',q} - f_{q,q'},$$

  and finally (45) guarantees that

$$0 \leq f''_{q',q} = f_{q',q} - f_{q,q'} - f'_{q,q'} \leq c_{q',q}.$$

Let us now prove the flow conservation. Let $q \in \mathcal{P}$.

- If $q \notin \mathcal{P}'$ and $q \neq s$, then for any $q' \in \sigma_{\mathcal{E}}(q)$ the definition of $\mathcal{E}'$ given in (28) guarantees that both (q, q') and $(q', q) \notin \mathcal{E}'$. Using (43), we obtain $f''_{q,q'} = f_{q,q'}$ and $f''_{q',q} = f_{q',q}$, for all $q' \in \sigma_{\mathcal{E}}(q)$, and therefore

$$\sum_{q' \in \sigma_{\mathcal{E}}(q)} f''_{q',q} = \sum_{q' \in \sigma_{\mathcal{E}}(q)} f_{q',q} = \sum_{q' \in \sigma_{\mathcal{E}}(q)} f_{q,q'} = \sum_{q' \in \sigma_{\mathcal{E}}(q)} f''_{q,q'}.$$

- If $q \in \mathcal{P}'$, the flow conservation constraint given by (11) for f and f' at q can be decomposed to provide

$$f_q + \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \notin \sigma_{\mathcal{E}'}(q)}} (f_{q',q} - f_{q,q'}) + \sum_{\substack{q' \in \sigma_{\mathcal{E}'}(q) \\ f_{q',q} > f_{q,q'}}} (f_{q',q} - f_{q,q'}) + \sum_{\substack{q' \in \sigma_{\mathcal{E}'}(q) \\ f_{q',q} \leq f_{q,q'}}} (f_{q',q} - f_{q,q'}) = 0$$

and

$$f'_q + \sum_{\substack{q' \in \sigma_{\mathcal{E}'}(q) \\ f_{q',q} > f_{q,q'}}} (0 - f'_{q,q'}) + \sum_{\substack{q' \in \sigma_{\mathcal{E}'}(q) \\ f_{q',q} \leq f_{q,q'}}} (f'_{q',q} - 0) = 0.$$

Summing these equalities and using (46), (43) and (45), we obtain

$$f''_q + \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \notin \sigma_{\mathcal{E}'}(q)}} (f''_{q',q} - f''_{q,q'}) + \sum_{\substack{q' \in \sigma_{\mathcal{E}'}(q) \\ f_{q',q} > f_{q,q'}}} (f''_{q',q} - f''_{q,q'}) + \sum_{\substack{q' \in \sigma_{\mathcal{E}'}(q) \\ f_{q',q} \leq f_{q,q'}}} (f''_{q',q} - f''_{q,q'}) = 0.$$

The latter corresponds to flow conservation constraint (11) at the node q for f''.

Altogether, we now know that f'' is a flow. We still need to show that it is a max-flow. The latter property is in fact trivially obtained since (44) and (43) guarantee that $f''_{q,t} = f_{q,t}$, for all $q \in \mathcal{P}$. Therefore, the value of f'' is equal to the value of f. Since f is a max-flow, this value is maximal and f'' is a max-flow. □

Proposition 4 *If Σ^+ is a minimum s-t cut in the graph $\mathcal{G}'$ defined in Section V, then the max-flow f'' is such that*

$$\forall q \in \sigma_{\mathcal{E}}(p), \qquad f''_{q,p} = 0.$$

As a consequence,

$$\forall q \in \sigma_{\mathcal{E}}(p), \qquad f''_{p,q} \geq f''_{q,p}.$$

Proof. Since f' is a max-flow in $\mathcal{G}'$ and Σ^+ is a min s-t cut in $\mathcal{G}'$, Ford-Fulkerson theorem guarantees that they have the same value. We therefore have

$$\begin{aligned} \mathrm{val}_{\mathcal{G}'}(f') = \mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right) &= \sum_{\substack{q' \in \Sigma^+ \\ q \notin \Sigma^+ \\ (q',q) \in \mathcal{E}'}} c'_{q',q} \\ &= \sum_{q \in \Sigma^-} c'_{p,q} \end{aligned} \tag{47}$$

Moreover, since f' is a flow, the total amount of flow entering and exiting Σ^+ are equal. Therefore, we have (see (11))

$$\sum_{q \in \Sigma^+} f'_q + \sum_{\substack{q' \in \Sigma^+ \\ q \notin \Sigma^+ \\ q \in \sigma_{\mathcal{E}'}(q')}} (f'_{q,q'} - f'_{q',q}) = 0.$$

Together with (8) and (29), this guarantees that

$$\mathrm{val}_{\mathcal{G}'}(f') = \sum_{q\in\Sigma^+} f'_q = \sum_{\substack{q'\in\Sigma^+\\ q\notin\Sigma^+\\ q\in\sigma_{\mathcal{E}'}(q')}} (f'_{q',q} - f'_{q,q'}) = \sum_{q\in\Sigma^-} (f'_{p,q} - f'_{q,p}).$$

Combined with (47), this provides

$$\sum_{q\in\Sigma^-} c'_{p,q} = \sum_{q\in\Sigma^-} f'_{p,q} - \sum_{q\in\Sigma^-} f'_{q,p}. \tag{48}$$

As a consequence,

$$\sum_{q\in\Sigma^-} f'_{q,p} = \sum_{q\in\Sigma^-} (f'_{p,q} - c'_{p,q}) \leq 0.$$

However, since for all $q \in \Sigma^-$, $f'_{q,p} \geq 0$, we finally obtain that

$$\forall q \in \Sigma^-, f'_{q,p} = 0.$$

Using (48) again, (23) and (33), this provides

$$\forall q \in \Sigma^-, f'_{p,q} = c'_{p,q} = f_{q,p} - f_{p,q}.$$

Therefore, using (23) and (45),

$$\forall q \in \Sigma^-, f''_{q,p} = 0. \tag{49}$$

Moreover, using (24) and (45), we also have

$$\forall q \in (\Sigma^+ \cap \sigma_{\mathcal{E}}(p)), f''_{q,p} = 0. \tag{50}$$

Combining (49) and (24), we finally obtain

$$\forall q \in \sigma_{\mathcal{E}}(p), f''_{q,p} = 0,$$

which concludes the proof. □

Proposition 5 *Let $\mathcal{G}$ be the graph defined in Section II, let B satisfy* (14) *and let us assume that $p \in \mathcal{P}$ is such that*

$$\forall q \in B_p, \quad c_q \geq 0 \quad \textit{and} \quad c_q \geq \sum_{\substack{q'\in\sigma_{\mathcal{E}}(q)\\ q'\notin B_p}} c_{q,q'}, \tag{51}$$

then, there exists a max-flow f in $\mathcal{G}$ such that

$$\forall q \in \sigma_{\mathcal{E}}(p), f_{p,q} \geq f_{q,p} = 0. \tag{52}$$

Proof. This is a straightforward consequence of Proposition 3, Proposition 2 and Proposition 4.

Indeed, if (51) holds, we know that for any max-flow f in $\mathcal{G}$ and any $S \subset \mathcal{P}'$

$$\sum_{q \in \Sigma^+ \setminus (S \cup \{p\})} \left[c_q + \sum_{q' \notin \Sigma^+} (f_{q',q} - f_{q,q'}) \right] \geq 0$$

and therefore, Proposition 2 guarantees that Σ^+ is a min s-t cut in $\mathcal{G}'$. Then, Proposition 3 guarantees that f'' is a max-flow in $\mathcal{G}$ and Proposition 4 guarantees that f'' satisfies (52). □

VI. A USELESS NODE

Throughout this section, we consider a graph $\mathcal{G}$ as constructed in Section II, a set B satisfying (14), a pixel $p \in \mathcal{P}$ satisfying (51) and a max-flow f in $\mathcal{G}$ satisfying (52).

The purpose of this section is to modify f so-that it remains a max-flow in $\mathcal{G}$ and satisfies

$$\forall q \in \sigma_{\mathcal{E}}(p), \quad f_{p,q} = f_{q,p} = 0.$$

The latter obviously implies that the node p is useless when computing the max-flow in $\mathcal{G}$.

Since the method for modifying f is analogous to the one used in Section V, we chose to use the same notations for the objects playing the same role. Beware not to confuse their definition.

First, we denote

$$\mathcal{P}' = B_p \quad , \Sigma^+ = B_p \setminus \{p\} \quad \text{and} \quad \Sigma^- = \{p\}. \tag{53}$$

In order to modify f, we build a graph $\mathcal{G}' = (\mathcal{P}', \mathcal{E}', c')$ where $\mathcal{E}'$and c' are defined below. We consider

$$\mathcal{E}' = \left(\mathcal{E} \cap \left(\Sigma^+ \times \Sigma^+\right)\right) \bigcup \left((\sigma_{\mathcal{E}}(p) \cap \Sigma^+) \times \Sigma^-\right) \bigcup \left(\{s\} \times \Sigma^+\right) \bigcup \{(p,t)\}. \tag{54}$$

We define the capacities c' by

$$c'_{q,q'} = c_{q,q'} - f_{q,q'} + f_{q',q} \qquad , \forall (q,q') \in \left(\mathcal{E} \cap \left(\Sigma^+ \times \Sigma^+\right)\right) \tag{55}$$

$$c'_{q,p} = f_{p,q} \qquad , \forall q \in (\sigma_{\mathcal{E}}(p) \cap \Sigma^+) \tag{56}$$

$$c'_{s,q} = c_q - f_q \qquad , \forall q \in \Sigma^+ \tag{57}$$

$$c'_{p,t} = f_p \tag{58}$$

As usual, in order to simplify the notations, we artificially set

$$c'_{q,q'} = 0 \quad , \forall (q,q') \in (\mathcal{P}' \times \mathcal{P}') \setminus \mathcal{E}', \tag{59}$$

and we write

$$c'_q = c'_{s,q} - c'_{q,t} \quad , \forall q \in \mathcal{P}'. \tag{60}$$

Notice first that, for any $S \subset \mathcal{P}'$, the value of the s-t cut $((S \cup \{s\}), (\mathcal{P}' \setminus S) \cup \{t\})$ in $\mathcal{G}'$ depends on whether $p \in S$ or $p \notin S$. If $p \in S$, we have

$$\mathrm{val}_{\mathcal{G}'}(S) = c'_{p,t} + \sum_{q \in (\Sigma^+ \setminus S)} c'_q + \sum_{\substack{q \in S \\ q' \in (\mathcal{P}' \setminus S)}} c'_{q,q'}.$$

Therefore, we trivially have using (55)-(60)

$$\mathrm{val}_{\mathcal{G}'}(S) \geq c'_{p,t} = f_p \quad , \text{if } p \in S. \tag{61}$$

Moreover, for any $S \subset \mathcal{P}'$, the value of the s-t cut $((S \cup \{s\}), (\mathcal{P}' \setminus S) \cup \{t\})$ in $\mathcal{G}'$ is given by

$$\mathrm{val}_{\mathcal{G}'}(S) = \sum_{q \in (\Sigma^+ \setminus S)} c'_q + \sum_{\substack{q \in S \\ q' \in (\mathcal{P}' \setminus S)}} c'_{q,q'} \quad , \text{if } p \notin S. \tag{62}$$

In particular, if $S = \Sigma^+$, we obtain using (56), the conservation of the flow f at p and (52) that

$$\begin{aligned} \mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right) &= \sum_{q \in (\Sigma^+ \cap \sigma_{\mathcal{E}'}(p))} c'_{q,p}, \\ &= \sum_{q \in (\Sigma^+ \cap \sigma_{\mathcal{E}}(p))} f_{p,q}, \\ &= f_p \ . \end{aligned} \tag{63}$$

The following proposition holds.

Proposition 6 *Let $\mathcal{G}'$ be the graph constructed in Section VI. For any $S \subset \mathcal{P}'$,*

- *if* $p \notin S$

$$\mathrm{val}_{\mathcal{G}'}(S) = \mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right) + \sum_{\substack{q\in S\\ q'\in(\Sigma^+\setminus S)}} c_{q,q'} + \sum_{q\in(\Sigma^+\setminus S)} \left[c_q + \sum_{q'\notin\mathcal{P}'} (f_{q',q} - f_{q,q'})\right], \quad (64)$$

- *if* $p \in S$

$$\mathrm{val}_{\mathcal{G}'}(S) \geq \mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right). \quad (65)$$

Proof. Notice first that, if $p \in S$, (65) is a straightforward consequence of (61) and (63). Let us assume from now on that $p \notin S$.

Since f is a flow, the total amount of flow entering and exiting $(\mathcal{P}' \setminus S)$ are equal (see (11)) and therefore, using (53)

$$f_p + \sum_{q\in(\Sigma^+\setminus S)} f_q + \sum_{\substack{q\in(\mathcal{P}'\setminus S)\\ q'\notin(\mathcal{P}'\setminus S)}} (f_{q',q} - f_{q,q'}) = 0.$$

Using (63), (57) and (60), we obtain

$$\mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right) + \sum_{q\in(\Sigma^+\setminus S)} (c_q - c'_q) + \sum_{\substack{q\in(\mathcal{P}'\setminus S)\\ q'\notin(\mathcal{P}'\setminus S)}} (f_{q',q} - f_{q,q'}) = 0.$$

Combined with (62), this becomes

$$\mathrm{val}_{\mathcal{G}'}(S) = \mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right) + \sum_{q\in(\Sigma^+\setminus S)} c_q + \sum_{\substack{q\in(\mathcal{P}'\setminus S)\\ q'\notin(\mathcal{P}'\setminus S)}} (f_{q',q} - f_{q,q'}) + \sum_{\substack{q\in S\\ q'\in(\mathcal{P}'\setminus S)}} c'_{q,q'}. \quad (66)$$

We now decompose the last term of the above equation using (55), (56) and (52) and write

$$\begin{aligned}
\sum_{\substack{q\in S\\ q'\in(\mathcal{P}'\setminus S)}} c'_{q,q'} &= \sum_{\substack{q\in S\\ q'\in(\Sigma^+\setminus S)}} (c_{q,q'} - f_{q,q'} + f_{q',q}) + \sum_{q\in S} f_{p,q} \\
&= \sum_{\substack{q\in S\\ q'\in(\Sigma^+\setminus S)}} c_{q,q'} - \sum_{\substack{q'\in S\\ q\in(\Sigma^+\setminus S)}} (f_{q',q} - f_{q,q'}) + \sum_{q'\in S} (f_{p,q'} - f_{q',p}) \\
&= \sum_{\substack{q\in S\\ q'\in(\Sigma^+\setminus S)}} c_{q,q'} - \sum_{\substack{q\in(\mathcal{P}'\setminus S)\\ q'\in S}} (f_{q',q} - f_{q,q'})
\end{aligned}$$

Combining the latter with (66), we finally obtain

$$\mathrm{val}_{\mathcal{G}'}(S) = \mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right) + \sum_{q\in(\Sigma^+\setminus S)} c_q + \sum_{\substack{q\in S\\ q'\in(\Sigma^+\setminus S)}} c_{q,q'} + \sum_{\substack{q\in(\mathcal{P}'\setminus S)\\ q'\notin\mathcal{P}'}} (f_{q',q} - f_{q,q'}).$$

Using (14), we remark that for any $q' \notin \mathcal{P}'$, $q' \notin \sigma_{\mathcal{E}}(p)$ and we can finally deduce that (64) holds for all $S \subset \mathcal{P}'$ such that $p \notin S$. □

As in Section V, we will from now on consider a max flow f' in the graph $\mathcal{G}'$ built in the current section. We also artificially extend the flow f' and set

$$f'_{q,q'} = 0, \text{ for all } (q,q') \in ((\mathcal{V}' \times \mathcal{V}') \setminus \mathcal{E}')\,. \tag{67}$$

Once again, the graph $\mathcal{G}'$ satisfies (1) and (3), therefore, as usual, we denote for simplicity

$$f'_q = f'_{s,q} - f'_{q,t} \quad ,\forall q \in \mathcal{P}'. \tag{68}$$

We are now going to combine f and f' in order to build a mapping $f'' : \mathcal{E} \to \mathbb{R}$ which will turn out to be a max-flow in $\mathcal{G}$ such that

$$f''_{p,q} = f''_{q,p} = 0 \qquad ,\forall q \in \sigma_{\mathcal{E}}(p).$$

As for $\mathcal{G}'$ and f', beware that the mapping f' is different in Section V and in the current section.

Let us begin with the definition of f''. We set

$$f''_q = f_q \qquad \forall q \notin \mathcal{P}' \tag{69}$$

$$f''_{q,q'} = f_{q,q'} \qquad \forall (q,q') \in \mathcal{E}, \text{ with } q \notin \mathcal{P}' \text{ or } q' \notin \mathcal{P}' \tag{70}$$

$$f''_q = f_q + f'_q \qquad \forall q \in \mathcal{P}' \tag{71}$$

$$f''_{q,q'} = (f_{q,q'} + f'_{q,q'}) - (f_{q',q} + f'_{q',q}) \qquad \forall (q,q') \in \left(\mathcal{E} \cap (\Sigma^+)^2\right) \text{ and } f_{q,q'} + f'_{q,q'} \geq f_{q',q} + f'_{q',q} \tag{72}$$

$$f''_{q,q'} = 0 \qquad \forall (q,q') \in \left(\mathcal{E} \cap (\Sigma^+)^2\right) \text{ and } f_{q,q'} + f'_{q,q'} < f_{q',q} + f'_{q',q} \tag{73}$$

$$f''_{p,q} = f_{p,q} - f'_{q,p} \qquad \forall q \in (\mathcal{P}' \cap \sigma_{\mathcal{E}}(p)) \tag{74}$$

$$f''_{q,p} = 0 \qquad \forall q \in (\mathcal{P}' \cap \sigma_{\mathcal{E}}(p)) \tag{75}$$

We also define

$$f''_{s,q} = \max(f''_q, 0) \quad \text{and} \quad f''_{q,t} = \max(-f''_q, 0) \quad ,\forall q \in \mathcal{P}. \tag{76}$$

Notice that the equation (69)-(76) permit to define $f''_{q,q'}$ for all $(q,q') \in \mathcal{E}$. Once again, we extend f'' outside $\mathcal{E}$ and set

$$f''_{q,q'} = 0, \text{ for all } (q,q') \in ((\mathcal{V} \times \mathcal{V}) \setminus \mathcal{E})\,.$$

The following proposition holds.

Proposition 7 *The mapping* $f'' : (\mathcal{V} \times \mathcal{V}) \to \mathbb{R}$ *is max-flow in* $\mathcal{G}$.

Proof. Notice first that, if f'' is a flow in $\mathcal{G}$ it is necessarily a max flow since, according to (51), $(\sigma_{\mathcal{E}}(t) \cap \mathcal{P}') = \emptyset$ and therefore, using (69), we always have $f''_{q,t} = f_{q,t}$, for all $q \in \sigma_{\mathcal{E}}(t)$. Therefore, the $\mathrm{val}_{\mathcal{G}}(f'') = \mathrm{val}_{\mathcal{G}}(f)$ and the latter is maximal in $\mathcal{G}$.

In order to show that f'' is a flow we first show that it satisfies the capacity constraints. Let $(q, q') \in \mathcal{E}$.

- If $q = s$ and $q' \notin B_p$ or if $q \notin B_p$ and $q' = t$, using (69) and (76), we know that

$$0 \leq f''_{q,q'} = f_{q,q'} \leq c_{q,q'} \quad \text{and} \quad 0 \leq f''_{q',q} = f_{q',q} \leq c_{q',q}.$$

- If $q \notin B_p$ or $q' \notin B_p$, using (70), we obtain again

$$0 \leq f''_{q,q'} = f_{q,q'} \leq c_{q,q'}.$$

- If $q = s$ and $q' \in \Sigma^+$, using (71) and (57), we get

$$0 \leq f''_{q,q'} = f_{s,q'} + f'_{s,q'} \leq c_{q,q'}.$$

- If $q = s$ and $q' = p$, using (71) and (58), we get

$$0 \leq f''_{q,q'} = f_{s,p} - f'_{p,t} \leq c_{q,q'}.$$

- If $(q, q') \in (\Sigma^+)^2$ and $f_{q,q'} + f'_{q,q'} \geq f_{q',q} + f'_{q',q}$, using (72) and (55), we obtain

$$0 \leq f''_{q,q'} = f_{q,q'} + f'_{q,q'} - f_{q',q} - f'_{q',q} \leq c_{q,q'} - f'_{q',q} \leq c_{q,q'}.$$

- If $(q, q') \in (\Sigma^+)^2$ and $f_{q,q'} + f'_{q,q'} < f_{q',q} + f'_{q',q}$, using (72), we trivially have

$$0 \leq f''_{q,q'} = 0 \leq c_{q,q'}.$$

- If $q = p$ and $q' \in (B_p \cap \sigma_{\mathcal{E}}(p))$, using (74) and (56), we get

$$0 \leq f''_{q,q'} = f_{p,q'} - f'_{q',p} \leq c_{q,q'}.$$

- If $q \in (B_p \cap \sigma_{\mathcal{E}}(p))$ and $q' = p$, then (75) trivially guarantees that

$$0 \leq f''_{q,q'} = 0 \leq c_{q,q'}.$$

In order to show the flow conservation constraints, we consider, from now on, $q \in \mathcal{P}$.

- If $q \notin \mathcal{P}'$, we have, using (69) and (70), we have $f''_{q,q'} = f_{q,q'}$ and $f''_{q',q} = f_{q',q}$, for all $q' \in \sigma_{\mathcal{E}}(q)$. Therefore,

$$\sum_{q \in \sigma_{\mathcal{E}}(q)} f''_{q',q} = \sum_{q \in \sigma_{\mathcal{E}}(q)} f_{q',q} = \sum_{q \in \sigma_{\mathcal{E}}(q)} f_{q,q'} = \sum_{q \in \sigma_{\mathcal{E}}(q)} f''_{q,q'}.$$

- If $q \in \Sigma^+$, expressing that the two flows f and f' are conserved at q, we obtain using (7) and (52)

$$f_q + \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \notin \mathcal{P}'}} (f_{q',q} - f_{q,q'}) + \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \in \Sigma^+}} (f_{q',q} - f_{q,q'}) + f_{p,q} = 0$$

and

$$f'_q + \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \in \Sigma^+}} (f'_{q',q} - f'_{q,q'}) \quad - f'_{q,p} = 0.$$

Summing those inequalities and using (70)-(75), we obtain

$$f''_q + \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \notin \mathcal{P}'}} (f''_{q',q} - f''_{q,q'}) + \sum_{\substack{q' \in \sigma_{\mathcal{E}}(q) \\ q' \in \Sigma^+}} (f''_{q',q} - f''_{q,q'}) + (f''_{p,q} - f''_{q,p}) = 0.$$

The latter expresses that f'' is conserved at the node q.

- If $q = p$, then using (71), (74) and (75) as well as (14) and (58), we obtain

$$\sum_{q \in \sigma_{\mathcal{E}}(p)} (f''_{q,p} - f''_{p,q}) = f_{s,p} - f'_{p,t} - \sum_{q \in (\mathcal{P}' \cap \sigma_{\mathcal{E}}(p))} (f_{p,q} - f'_{q,p}).$$

Using that $f_{p,t} = 0$ (see (51), (4) and (3)), $f'_{s,p} = 0$ (see (54) and (59)), $f_{q,p} = 0$ (see (52)) and $f'_{p,q} = 0$ (see (54) and (59)), we obtain

$$\sum_{q \in \sigma_{\mathcal{E}}(p)} (f''_{q,p} - f''_{p,q}) = (f_{s,p} - f_{p,t}) + (f'_{s,p} - f'_{p,t}) - \sum_{q \in (\mathcal{P}' \cap \sigma_{\mathcal{E}}(p))} \left[(f_{p,q} - f_{q,p}) + (f'_{p,q} - f'_{q,p}) \right].$$

Simplifying, we finally obtain

$$\begin{aligned} \sum_{q \in \sigma_{\mathcal{E}}(p)} (f''_{q,p} - f''_{p,q}) &= \sum_{q \in \sigma_{\mathcal{E}}(p)} (f_{q,p} - f_{p,q}) + \sum_{q \in \sigma_{\mathcal{E}}(p)} (f'_{q,p} - f'_{p,q}), \\ &= 0, \end{aligned}$$

since the two flows f and f' are conserved at p.

This concludes the proof. □

Proposition 8 *If Σ^+ is a minimum s-t cut in the graph $\mathcal{G}'$ defined in Section VI, then the max-flow f'' is such that*

$$\forall q \in \sigma_{\mathcal{E}}(p), \qquad f''_{q,p} = f''_{p,q} = 0.$$

As a consequence, removing the node p from the graph $\mathcal{G}$ does not modify its maximal flow value.

Proof. If Σ^+ is a minimum s-t cut in the graph $\mathcal{G}'$ defined in Section VI, then Ford-Fulkerson theorem, (63) and (54) guarantee that

$$f_p = \mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right) = \mathrm{val}_{\mathcal{G}'}\left(f'\right) = \sum_{q\in\Sigma^+} f'_q.$$

Now, since the total amount of flow f' entering and exiting Σ^+ are equal, we obtain, using (54), that

$$f_p = \sum_{q\in(\sigma_{\mathcal{E}}(p)\cap\Sigma^+)} f'_{q,p}.$$

Using that the flow f' is preserved at p and (54), we finally get

$$f_p = f'_{p,t}.$$

Using (71), (68) and (67) this yields

$$f''_p = f_p - f'_{p,t} = 0,$$

which, using (76), provides

$$f''_{s,p} = f''_{p,t} = 0.$$

Together with (75), this guarantees that

$$\text{for all } q \in \sigma_{\mathcal{E}}(p), \quad f''_{q,p} = 0. \tag{77}$$

Expressing the flow conservation constraint at p for f'', we deduce from (77) that

$$\sum_{q\in\sigma_{\mathcal{E}}(p)} f''_{p,q} = \sum_{q\in\sigma_{\mathcal{E}}(p)} f''_{q,p} = 0,$$

which guarantees that

$$\text{for all } q \in \sigma_{\mathcal{E}}(p), \quad f''_{p,q} = 0,$$

since $f''_{p,q} \geq 0$, for all $q \in \sigma_{\mathcal{E}}(p)$.

Together with (77), this concludes the proof. □

We can now conclude with the following proposition.

Proposition 9 *Let $\mathcal{G}$ be the graph defined in Section II, let B satisfy* (14) *and let us assume that $p \in \mathcal{P}$ satisfies* (51)*. Then, there exists a max-flow f in $\mathcal{G}$ such that*

$$\forall q \in \sigma_{\mathcal{E}}(p), \qquad f_{p,q} = f_{q,p} = 0. \tag{78}$$

As a consequence, removing the node p from the graph $\mathcal{G}$ does not modify its maximal flow value.

Proof. This is a straightforward consequence of Proposition 5, Proposition 6, Proposition 7 and Proposition 8.

Indeed, if (51) holds, we know that there is max-flow f in $\mathcal{G}$ satisfying (52). Therefore, using the notations of Section VI, we know that for any $S \subset \mathcal{P}'$ such that $p \notin S$

$$\sum_{q \in \Sigma^+ \setminus S} \left[c_q + \sum_{q' \notin \mathcal{P}'} (f_{q',q} - f_{q,q'}) \right] \geq 0.$$

Therefore, for $\mathcal{G}'$ as defined in Section VI, Proposition 6 guarantees that for any $S \subset \mathcal{P}'$

$$\mathrm{val}_{\mathcal{G}'}(S) \geq \mathrm{val}_{\mathcal{G}'}\left(\Sigma^+\right),$$

and therefore Σ^+ is a min s-t cut in $\mathcal{G}'$. Then, Proposition 7 guarantees that f'' is a max-flow in $\mathcal{G}$ and Proposition 8 guarantees that f'' satisfies (78). □

VII. NUMERICAL EXPERIMENTS

In this section, we evaluate the performance of the test (16) against standard graph cuts (SGC) in terms of speed and memory for reducing graphs involved in binary image segmentation. We also compare the relative reduced graph sizes obtained from the tests (16) and [9]. The relative reduced graph size of a graph $\mathcal{G}$ is defined by

$$\rho = \frac{\sharp \mathcal{V}}{\sharp \mathcal{V}'} \times 100.$$

In the experiments, B corresponds to a square window of size $(2r+1)$ centered at the origin. Moreover, we use the image segmentation model described in [14] in connectivity 1 [2]. Let us now briefly remind this model. Consider an image $I : \mathcal{P} \rightarrow [0,1]^c$ as a function, mapping each pixel $p \in \mathcal{P}$ to a vector $I_p \in [0,1]^c$. For any pixel $p \in \mathcal{P}$, the data term $E_p(.)$ is defined as

$$\begin{cases} E_p(1) = -log\ \mathbb{P}(I_p \mid p \in \mathcal{O}), \\ E_p(0) = -log\ \mathbb{P}(I_p \mid p \in \mathcal{B}). \end{cases}$$

where $\mathcal{O}$ and $\mathcal{B}$ denote respectively object and background seeds given by the user. The distribution of the object and the background are estimated using normalized histograms with a number of bins respectively equal to 256 and 50 for grayscale and color images. For any pair $(p,q) \in (\mathcal{P} \times \mathcal{P})$, the smoothness term $E_{p,q}(.)$ is defined as a contrast-sensitive Ising model

$$E_{p,q}(u_p, u_q) = \begin{cases} 0 & \text{if } u_p = u_q, \\ \frac{1}{\|p-q\|_2} exp\Big(- \frac{\|I_p - I_q\|_2^2}{2\sigma^2} \Big) & \text{otherwise,} \end{cases}$$

where $\|.\|_2$ denotes the Euclidean norm (either in $\mathbb{R}^d$ or $\mathbb{R}^c$) and σ is a parameter. The experiments are performed on an Athlon Dual Core 6000+ 3GHz with 2Gb RAM using the max-flow algorithm of [10]. The running times include the graph construction, the max-flow computation as well as the the construction of the solution. Times are averaged over 10 runs.

Let us now describe the experimental procedure for segmenting a subset of the images of [9]. For each image, the seeds and the model parameters are manually optimized for getting the best segmentation. Using these seeds and parameters, a reference segmentation is computed with SGC. Afterwards, a second segmentation is computed with the test (16) using the same seeds and parameters. The differences between both segmentations are assessed using the Hausdorff distance. We also measure the difference $\Delta\rho^*$ between the minimal reduced graph sizes (denoted by ρ^*) respectively obtained from the tests (16) and [9]. In words, the test [9] is more efficient than the test (16) when $\Delta\rho^* > 0$ and conversely. The results are summarized in Table I and illustrated in Figure 3.

As the test [9], the test (16) globally outperforms SGC in terms of memory while keeping a Hausdorff distance null. SGC fail to segment some large volumes while the test (16) permits to segment them in a reasonable time. Similarly to the test [9], reduced graphs are larger when the

[2] This term corresponds to 8 neighbors in 2D images and 26 neighbors in 2D+t and 3D images.

Volume name	**Size**	**SGC**		**Test** (16)		ρ^* (%)	$\Delta\rho^*$ (%)
		Time	**Memory**	**Time**	**Memory**		
zen-garden-c	481 ×321	0.22	22.90 Mb	1.23	23.39 Mb	90.24	+0.51
red-flowers-c	481 ×321	0.20	22.90 Mb	1.06	10.40 Mb	46.26	-22.96
book	3012 ×2048	7.59	917.26 Mb	42.48	78.95 Mb	7.91	+0.27
cells-z	512 ×512	0.48	38.91 Mb	2.08	23.39 Mb	61.65	-12.74
interview-man-2c	426 ×240 ×180	**OM**	7.55 Gb	1018.84	228.44 Mb	3.21	0.0
plane-take-off-c	492 ×276 ×180	**OM**	10.03 Gb	1346.40	532.00 Mb	6.09	+0.11
fluorescent-cell-c	478 ×396 ×121	**OM**	9.39 Gb	1490.55	514.00 Mb	5.88	0.0
ct-thorax	245 ×245 ×151	**OM**	3.71 Gb	493.23	771.00 Mb	17.30	0.0
cells	230 ×230 ×57	9.40	1.23 Gb	156.79	771.00 Mb	59.38	-8.0
brain-p3	181 ×217 ×181	**OM**	2.91 Gb	381.72	771.00 Mb	24.22	+0.16

TABLE I: SGC are compared to RGC in terms of speed (in secs) and memory for segmenting 2D (top), 2D+t (middle) and 3D (bottom) images. Labels **OM** and **NSR** resp. stands for "Out of Memory" and "No Segmentation Reference". Color images names are suffixed by "c".
table

amount of regularization is large. Indeed, since only capacities to terminal nodes are multiplied by β, the test (16) becomes harder to satisfy when β diminishes and conversely. This is also the case for noisy images since a lot a nodes inside B are connected to opposite terminals (see e.g. images "zen-garden-c" and "cells-z"). An ideal situation therefore consists as in [9] of large area of nodes linked to the same terminal and separated by smooth borders.

While SGC remain faster than the test (16), the latter is globally less efficient than the test [9] with a negative average $\Delta\rho^*$ over all images. This least performance is strenghtened when the amount of regularization is large. Indeed, in such a situation, the test [9] can be relaxed when varying the window radius r unlike the test (16). However, when the amount of regularization is of moderate level, the memory gains are almost the same. We have also measured no differences between the segmentation obtained with the test (16) and the test [9]. This clearly demonstrates that the test [9] is an heuristic achieving very good results. In words, the exactness of the test (16) is at the expense of a larger computational cost.

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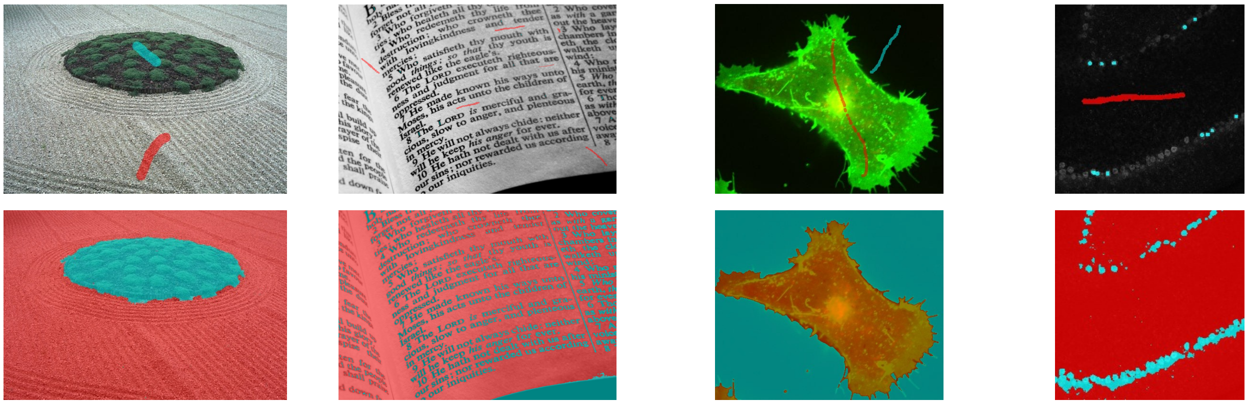

"zen-garden-c" (90.24%) "book" (7.91%) "fluorescent-cell-c" (5.88%) "cells" (59.38%)

Fig. 3: Segmentations (top row) and reduced graphs (bottom row) results. Reduced graphs are superimposed in yellow by transparency. Reduced graph sizes are indicated in parenthesis.
figure

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