Independence of hyperlogarithms over function fields via algebraic combinatorics

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Outline

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Hyperlogarithms

Definition (1928, Lappo-Danilevski)

Let $a_0, \ldots a_k \in \mathbb{C}$. Then

$$L(a_{i_n}, \ldots, a_{i_0}|\gamma) = \int_{z_0}^{z} \int_{z_0}^{s_n} \ldots \int_{z_0}^{s_1} \frac{ds_0}{s_0 - a_{i_0}} \ldots \frac{ds_n}{s_n - a_{i_n}}$$

with $\gamma : z_0 \rightsquigarrow z$ a path such that

$$a_{j_i} \notin \gamma \text{ and } s_i \in \gamma, \forall i \in \{1, \ldots, n\}.$$

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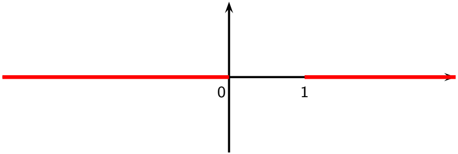
- Largest field on which these functions are independent ?
- If no linear independence, describe the relations.

Polylogarithms

$X = \{x_0, x_1\}$. Let $\mathscr{C} = \mathbb{C} \setminus (]-\infty, 0[\cup]1, +\infty[)$.



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Definition

$\forall z \in \mathscr{C}$,

$$\mathrm{Li}_{x_1}(z) = -\ln(1-z) \text{ and } \mathrm{Li}_{x_0}(z) = \ln(z).$$

$$\mathrm{Li}_{x_0 w}(z) = \int_0^z \frac{dt}{t}\mathrm{Li}_w(t); \qquad \mathrm{Li}_{x_1 w}(z) = \int_0^z \frac{dt}{1-t}\mathrm{Li}_w(t).$$

$w = x_0^{s_1-1}x_1 \ldots x_0^{s_k-1}x_1 \leftrightarrow \mathbf{s} = (s_1, \ldots, s_k)$.

$$\mathrm{Li}_w(z) = \mathrm{Li}_{\mathbf{s}}(z) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_k^{s_k}}.$$

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$\mathbb{C}$-linear independence [Van der Hoeven, Hoang Ngoc Minh, Petitot, 2000].

Coloured Polylogarithms

Definition

$$\mathrm{Li}_{\mathbf{b},\mathbf{s}}(z) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{b_1^{n_1} \ldots b_k^{s_k}}{n_1^{s_1} \ldots n_k^{s_k}} z^{n_1}$$

with $\mathbf{s} = (s_1, \ldots, s_k)$, $\mathbf{b} = (b_1, \ldots, b_k)$.

Linear independence over $\mathbb{C}\left[z, \frac{1}{z}, \left(\frac{1}{\rho_i^{-1} - z}\right)_{i=0,\ldots,n}\right]$ where the $(\rho_i)_i$'s are the n^{th} roots of 1 [Hoang Ngoc Minh, 2004].

Encoding integrals with words

Let $u_i(z)$, $0 \leq i \leq n$, be n inputs analytical on $\mathscr{C}$ and $X = \{x_1, \ldots, x_n\}$.

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Recursive definition of the iterated integrals $\alpha_{z_0}^{z}(w)$ for $w \in X^*$ and $z \in \mathscr{C}$:

$$\alpha_{z_0}^{z}(1_{X^*}) = 1;$$

$$\alpha_{z_0}^{z}(x_i) = \int_{z_0}^{z} u_i(s)ds,\ x_i \in X;$$

$$\alpha_{z_0}^{z}(x_i w) = \int_{z_0}^{z} u_i(s)ds\, \alpha_{z_0}^{s}(w),\ x_i \in X,\ w \in X^*.$$

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Then if $u_i(z) = \dfrac{1}{z - a_i} \leftrightarrow x_i$,

$$\alpha_{z_0}^{z}(x_{j_0} \dots x_{j_n}) = L(a_{j_n}, \dots, a_{j_0}|\gamma)$$

with γ a path in $\mathscr{C}$ avoiding the a_i's.
Generating series of hyperlogarithms :

$$L(z) := \sum_{w \in X^*} \alpha_{z_0}^{z}(w)w.$$

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General idea : Derivating term by term L, we obtain the following non commutative differential equation :

$$\frac{d}{dz}L(z) = M(z)L(z),\ \text{with } M(z) = \sum_{x_i \in X} u_i(z)x_i.$$

Outline

Non commutative polynomials and series

Let R be a (commutative) ring (with unit).

- $R\langle X\rangle$: non commutative polynomials ;
- $R\langle\langle X\rangle\rangle$: non commutative series ;
- For $S \in R\langle\langle X\rangle\rangle$, $\langle S|w\rangle$ is the coefficient of S on the word w :

$$S = \sum_{w\in X^*} \langle S|w\rangle w.$$

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Special case of differential algebra : $R = (\mathfrak{A}, \mathrm{d})$, a commutative differential algebra over (a ring) k.

- differential : $\forall\, a, b \in \mathfrak{A},\ \mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}(b)$;
- d is linear over k ;
- we require (for the following theorems) that $\ker(\mathrm{d}) = k$ (the only constants for the derivation are the elements of the ground field k).

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- d is linear over k ;
- $\ker(\mathrm{d}) = k$.

We can extend in a unique way d to $\mathfrak{A}\langle\langle X\rangle\rangle$:

$$\forall S \in \mathfrak{A}\langle\langle X\rangle\rangle,\ \mathrm{d}(S) = \sum_{w \in X^*} \mathrm{d}(\langle S|w\rangle)w.$$

Integrator

Let $M \in \mathfrak{A}_{=1}\langle\langle X\rangle\rangle$ $(M(z) = \sum_{x\in X} u_x(z)x)$ and $z_0 \in \mathbb{C}$.

We define the integrator H_{z_0} :

$$H_{z_0} : \begin{array}{c} \mathfrak{A}\langle\langle X\rangle\rangle \to \mathfrak{A}_{\geq 1}\langle\langle X\rangle\rangle \\ S \mapsto H_{z_0}[S] = \int_{z_0}^{z} M(s)S(s)ds \end{array}$$

with

$$\int_{z_0}^{z} T(s)ds = \sum_{w\in X^*} \left(\int_{z_0}^{z} \langle T|w\rangle ds\right) w, \quad \forall T \in \mathfrak{A}\langle\langle X\rangle\rangle.$$

Iterated integrals

Since $\forall S,\ H_{z_0}^{n}[S] \in \mathfrak{A}_{\geq n}\langle\langle X\rangle\rangle$,

$$\langle H_{z_0}^{n}[S]\,|w\rangle \neq 0 \text{ only for } n \leq |w|.$$

Therefore, we can define the sum

$$\sum_{w\in X^*}\sum_{n\geq 0}\langle H_{z_0}^{n}[S]\,|w\rangle w = H_{z_0}^{*}[S] = \sum_{n\geq 0} H_{z_0}^{n}[S]\,.$$

Link with hyperlogarithms : With $u_i(z) = \dfrac{1}{z-a_i}$, $\forall i$,

$$H_{z_0}^{*}[1] = L(z) = \sum_{w\in X^*}\alpha_{z_0}^{z}(w)w$$

and

$$\langle H_{z_0}^{*}[1]\,|w\rangle = \alpha_{z_0}^{z}(w).$$

(Non commutative) Differential equation

It is clear that

$$H_{z_0}^* = 1 + H_{z_0} H_{z_0}^*$$

Therefore, $\forall\, S \in \mathfrak{A}\langle\langle X \rangle\rangle$ such that $dS = 0$ (constant series),

$$\mathrm{d}\left(H_{z_0}^*\,[S]\right) = \mathrm{d}\left(S + H_{z_0}\left(H_{z_0}^*\,[S]\right)\right) = MH_{z_0}^*\,[S]\,,$$

and $H_{z_0}^*\,[S]$ satisfies the (non commutative) differential equation

$$\mathrm{d}T = MT.$$

Outline

Theorem 1 : Linear independence

Let $\mathfrak{C}$ be a differential subfield of $\mathfrak{A}$ (i.e. $\mathrm{d}(\mathfrak{C}) \subset \mathfrak{C}$).
We suppose that $T \in \mathfrak{A}\langle\langle X\rangle\rangle$ is a solution of the differential equation

$$\mathrm{d}T = MT; \langle T|1_{X^*}\rangle = 1$$

where M is a homogeneous series of degree 1 : $M = \sum_{x\in X} u_x x \in \mathfrak{C}_{\geq 1}\langle\langle X\rangle\rangle$.

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The following conditions are equivalent :

i) The family $(\langle T|w\rangle)_{w \in X^*}$ of coefficients of T is free over $\mathfrak{C}$.

ii) The family of coefficients $(\langle T|y\rangle)_{y \in X \cup \{1_{X^*}\}}$ is free over $\mathfrak{C}$.

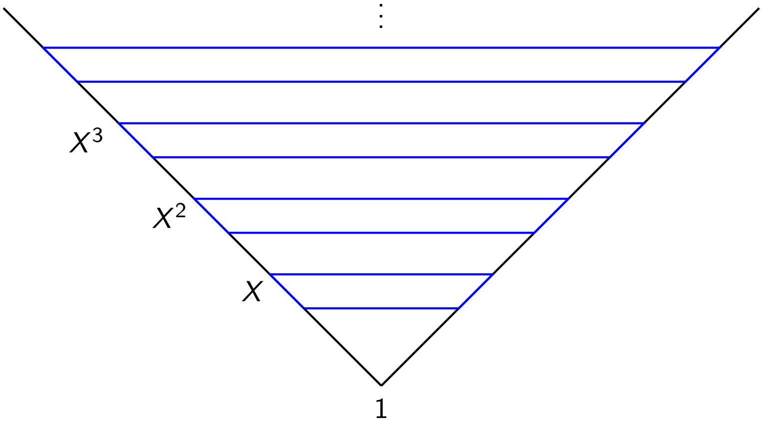
iii) The family $(u_x)_{x \in X}$ is such that, for $f \in \mathfrak{C}$ and $\alpha_x \in k$

$$\mathrm{d}(f) = \sum_{x \in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0).$$

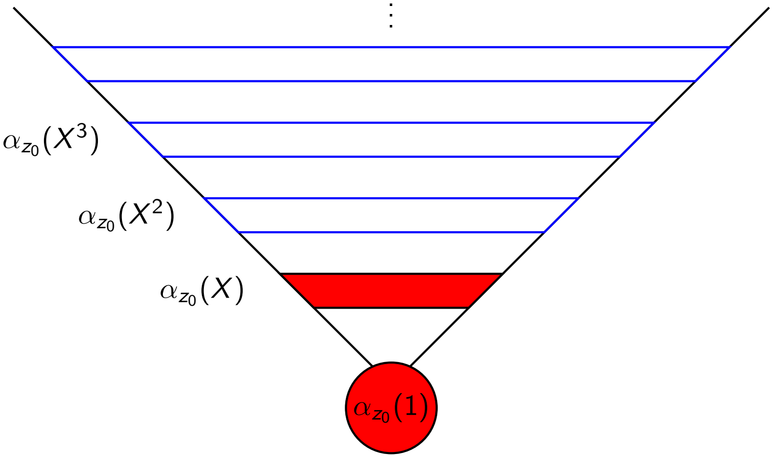
iv) The family $(u_x)_{x \in X}$ is free over k and

$$\mathrm{d}(\mathfrak{C}) \cap \mathrm{span}_k \left((u_x)_{x \in X} \right) = \{0\}.$$

"Slices" of the free monoid



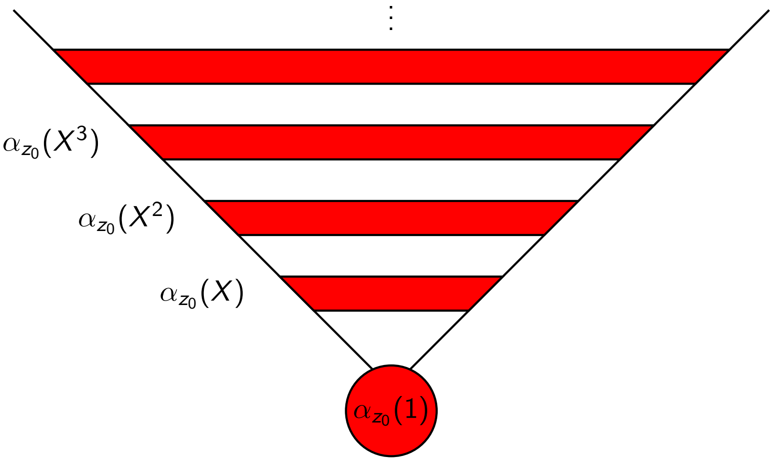
Linear independence of the "first triangle"



Linear independence of the "first triangle"

$\Rightarrow$

Linear independence of the whole triangle



Field of germs

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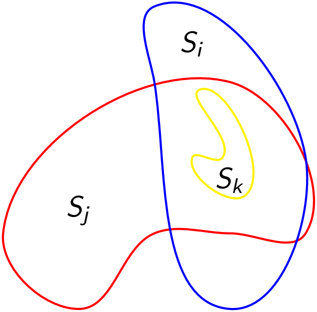
- V : connected, simply connected, analytic variety : $\mathbb{C}\setminus\{a_1, \ldots, a_n\}$;
- $\mathfrak{B}$: a filter basis of V of open connected subsets of V :

$$(\mathbb{C}\setminus(a_j)_{j\subset\mathscr{J}})_{\mathscr{J}\in\mathscr{I}};$$



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- $\mathfrak{B}$: $(\mathbb{C}\setminus(a_j)_{j\subset\mathscr{J}})_{i\in\mathscr{I}}$;
- A correspondence C such that

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- Compatibility with restrictions : $\forall U,W\in\mathfrak{B},\ W\subset U$,

$$\mathrm{rest}_{WU}(C[U])\subset C[W].$$

Theorem 2 : Application to hyperlogarithms

Let

- $M = \sum_{i=1}^{n} \frac{\lambda_i}{z - a_i} x_i$ be a multiplier on X with all $\lambda_i \neq 0$,
- S be any regular solution of

$$\frac{d}{dz} S(z) = M(z) S(z).$$

Then, let C be a differential field of functions defined on V which does not contain linear combinations of logarithms on any domain but which contains z and the constants (as, for example, the rational functions).
If $U \in \mathfrak{B}$ and $P \in C[U]\langle X \rangle$, one has

$$\langle S|P \rangle = 0 \Longrightarrow P = 0.$$

Linear independence on $C[U]$.

Application : Hyperlogarithms

Corollary of the previous theorem[1] **:**

R denotes the ring of functions that can be analytically extended to some $V \cup U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n}$ (U_{a_i} open neighborhood of a_i) and have non-essential singularities at these points.

Then the set of hyperlogarithms $(\langle S|w\rangle)_{w \in X^*}$ is a a set of linearly independent functions over R.

[1] **Independence of hyperlogarithms over function fields via algebraic combinatorics**, M. D., G. H. E. Duchamp, H. N. Minh and A. Solomon, *CAI 2011*, LNCS 6742, pp. 127–139. Springer, Heidelberg (2011)

Conclusion and perspectives

Conclusion :

- New and simpler proof of known results (without monodromy) ;
- Generalization of these results to a wider class of algebras.

Perspectives :

- Find the maximal field on which the linear independence is preserved ;
- Write a program that
  - tests the linear (in)dependence of functions over a given differential ring ;
  - gives the relations between the coefficients if necessary.

Thank you for your attention!