

How to compute Selberg-like Integrals?

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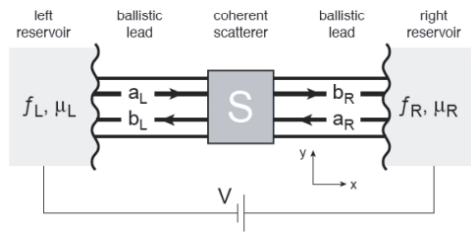
Séminaire **Combinatoire et Algorithmes**
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Outline

- 1 Physical motivation of the problem and history of Selberg's integral
 - Physical motivation
 - History
- 2 Some theory about symmetric functions
 - λ -ring structure
 - Deformation of the usual scalar product
 - Jack and Macdonald polynomials
- 3 Simplifications and algorithm
- 4 Computation and asymptotics of the power sums - Selberg integral
 - Decomposition of p_μ in the Jack basis
 - Conjectures

The physical problem

- Quantum transport in chaotic cavities :



- Unitary scattering matrix** S (relating the wave functions of incoming and outgoing electrons).
- Parametrisation of S :

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

(transmission t and reflection r matrices).

Random Matrix Theory Approach

- Important parameters of the system are computed with the **eigenvalues x_i of tt^\dagger** . For example :

$$\sum_{i=1}^n x_i \text{ (conductance);}$$

$$\sum_{i=1}^n x_i(1 - x_i) \text{ (shot noise).}$$

- S can be fairly approximated by a **random matrix** belonging to a suitable ensemble (whose choice depends on the system).

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- S can be fairly approximated by a **random matrix** belonging to a suitable ensemble (whose choice depends on the system).
- **Unitary constraint**: the joint probability density for the eigenvalues x_i of the matrix is of the following form (Beenakker):

$$P(x_1, \dots, x_N) = \frac{1}{\mathfrak{N}} \prod_{i < j} (x_i - x_j)^{2\kappa} \prod_{i=1}^N x_i^{\alpha-1}.$$

Integrals and Asymptotics

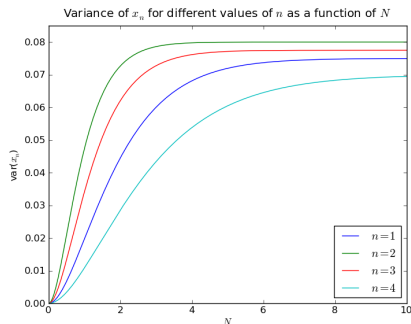
The physical quantities of interest are of the form :

$$\langle f \rangle_{a,b,\kappa}^N = \frac{1}{N!} \int_{[0,1]^N} f(x_1, \dots, x_N) \prod_{i < j} (x_i - x_j)^{2\kappa} \prod_{i=1}^N x_i^{a-1} (1-x_i)^{b-1} dx_i$$

for f a multivariate polynomial.

Two goals :

- Computation of the previous integral for several classes of functions.
- Computation of the asymptotics for $N \rightarrow \infty$ (simulating large systems).



(Short) History of Selberg's integrals 1/2

- **First appearance** : 1941 in Selberg's paper "Über einen Einsatz von A. Gelfond" (*Archiv for Matematik og Naturvidenskab*), with the change of variables $x_i = s_i/(1 + s_i)$ and integration over $[0, +\infty[$.
- **First proof** : 1944 in Selberg's paper "Bemerkninger om et multipelt integral" (*Norsk Matematisk Tidsskrift*).

$$\bullet \int_0^1 \dots \int_0^1 \prod_{i < j} (x_i - x_j)^{2\kappa} \prod_{i=1}^n x_i^{a-1} (x_i - 1)^{b-1} dx_i =$$

$$\prod_{j=0}^{n-1} \frac{\Gamma(a + j\kappa)\Gamma(1 + (j + 1)\kappa)}{\Gamma(a + b + (n + j - 1)\kappa)\Gamma(1 + \kappa)}.$$

(Short) History of Selberg's integrals 2/3

- **Aomoto's integral (1987) :**

$$\int_0^1 \dots \int_0^1 \prod_{i < j} (x_i - x_j)^{2\kappa} \left(\prod_{i=1}^k x_i \right) \prod_{i=1}^n x_i^{a-1} (x_i - 1)^{b-1} dx_i =$$

$$\prod_{j=0}^{n-1} \frac{\Gamma(a + j\kappa)\Gamma(1 + (j + 1)\kappa)}{\Gamma(a + b + (n + j - 1)\kappa)\Gamma(1 + \kappa)} \prod_{j=1}^k \frac{a + (n - j)\kappa}{a + b + (2n - j - 1)\kappa}.$$

(Short) History of Selberg's integrals 3/3

Fundamental result (Kadell, 1988) for Jack $P_{\lambda}^{(1/\kappa)}$ polynomials

$$\langle P_{\lambda}^{(1/\kappa)} \rangle_{a,b,\kappa}^N = \prod_{i < j} \frac{\Gamma[\lambda_i - \lambda_j + \kappa(j - i + 1)]}{\Gamma[\lambda_i - \lambda_j + \kappa(j - i)]} \prod_{i=1}^N \frac{\Gamma[\lambda_i + a + \kappa(N - i)] \Gamma[b + \kappa(N - i)]}{\Gamma[\lambda_i + a + b + \kappa(2N - i - 1)]}.$$

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- Integral for 2 Jack polynomials (Kadell, 1992) :

$$\begin{aligned} \langle P_\lambda^{(1/\kappa)} P_\mu^{(1/\kappa)} \rangle_{a,b,\kappa}^N &= \prod_{i,j=1}^N \frac{1}{(a + (2N - i - j)\kappa + \lambda_i + \mu_j)_\kappa} \\ & \prod_{i < j} \frac{\Gamma[\lambda_i - \lambda_j + \kappa(j - i + 1)]}{\Gamma[\lambda_i - \lambda_j + \kappa(j - i)]} \prod_{i < j} \frac{\Gamma[\mu_i - \mu_j + \kappa(j - i + 1)]}{\Gamma[\mu_i - \mu_j + \kappa(j - i)]}. \end{aligned}$$

Aim

I present¹ an algorithm to compute

$$\langle f \rangle_{a,b,\kappa,N}^{\#} := \frac{\langle f \rangle_{a,b,\kappa}^N}{\langle 1 \rangle_{a,b,\kappa}^N}$$

for any multivariate polynomial f , and explicit results for the power sums and the asymptotics of the integral for these functions.

¹ **Asymptotics of Selberg-like integrals: The unitary case and Newton's interpolation formula.** Christophe Carré, Matthieu Deneufchâtel, Jean-Gabriel Luque, Pierpaolo Vivo. *Journal of Mathematical Physics* 51, 12, (2010).

How to compute Selberg-like integrals? Matthieu Deneufchâtel, in *Proceedings of the 13th Mons Theoretical Computer Science Days, Amiens, 2010.*

Notations

- Sym the algebra of symmetric functions;
- S_i the complete functions of degree i : sum over all monomials of total degree i in x_1, \dots, x_N ;
- p_i the power sums : $p_i = \sum_{j=1}^N x_j^i$;
- σ_z the Cauchy kernel : $\sigma_z(\mathbb{X}) := \prod_{x \in \mathbb{X}} \frac{1}{1 - xz}$.

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Property :
$$\sigma_z(\mathbb{X}) = \sum_i S_i(\mathbb{X}) z^i.$$

λ -ring structure

Sym is endowed with a **λ -ring structure** :

- **Sum** of two alphabets :

$$\sigma_z(\mathbb{X} + \mathbb{Y}) = \sigma_z(\mathbb{X})\sigma_z(\mathbb{Y}) = \sum_i S_i(\mathbb{X} + \mathbb{Y})z^i.$$

- **Multiplication** by a constant :

$$\sigma_z(u\mathbb{X}) = \sigma_z^u(\mathbb{X}), \forall u \in \mathbb{C}.$$

- **Product** of two alphabets :

$$\sigma_1(\mathbb{X}\mathbb{Y}) = \sum_i S_i(\mathbb{X}\mathbb{Y}) = \prod_{x \in \mathbb{X}} \prod_{y \in \mathbb{Y}} \frac{1}{1 - xyt}.$$

Encoding operations that are defined on generating functions.

\langle, \rangle denotes the usual scalar product on Sym such that

$$\langle p_\mu, p_\nu \rangle = z_\mu \delta_{\mu, \nu}$$

with $p_\mu = \prod_i^{l(\mu)} p_{\mu_i}$.

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with $\mathbf{p}_\mu = \prod_i^{\ell(\mu)} \mathbf{p}_{\mu_i}$. We consider two deformations:

- a **one parameter deformation** $\langle, \rangle_{\frac{1}{\kappa}}$ such that

$$\langle \mathbf{p}_\mu, \mathbf{p}_\lambda \rangle_{\frac{1}{\kappa}} = z_\lambda \left(\frac{1}{\kappa} \right)^{\ell(\lambda)} \delta_{\mu, \lambda}.$$

- the canonical **two parameter deformation** $\langle, \rangle_{q,t}$ such that

$$\langle \mathbf{p}_\lambda, \mathbf{p}_\mu \rangle_{q,t} = z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \delta_{\lambda, \mu} = z_\lambda(q, t) \delta_{\lambda, \mu}.$$

with $z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!$.

Jack polynomials:

- defined as the unique family orthogonal *w.r.t.* $\langle, \rangle_{\frac{1}{\kappa}}$ and such that

$$P_{\lambda}^{\left(\frac{1}{\kappa}\right)}(\mathbb{X}) = m_{\lambda}(\mathbb{X}) + \sum_{\mu \leq \lambda} u_{\lambda, \mu}^{\kappa} m_{\mu}(\mathbb{X}).$$

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- obtained by applying Gram-Schmidt orthogonalization process to the Schur basis for $\langle, \rangle_{\frac{1}{\kappa}}$ beginning with 1^n .

Algorithmic construction of a basis of symmetric functions of degree 3:

$$P_{111}^{(\frac{1}{\kappa})} = S_{111}$$

$$P_{21}^{(\frac{1}{\kappa})} = S_{21} - \frac{\langle S_{21}, P_{111}^{(\frac{1}{\kappa})} \rangle_{\frac{1}{\kappa}}}{\langle P_{111}^{(\frac{1}{\kappa})}, P_{111}^{(\frac{1}{\kappa})} \rangle_{\frac{1}{\kappa}}} P_{111}^{(\frac{1}{\kappa})}$$

$$P_3^{(\frac{1}{\kappa})} = S_3 - \frac{\langle S_3, P_{21}^{(\frac{1}{\kappa})} \rangle_{\frac{1}{\kappa}}}{\langle P_{21}^{(\frac{1}{\kappa})}, P_{21}^{(\frac{1}{\kappa})} \rangle_{\frac{1}{\kappa}}} P_{21}^{(\frac{1}{\kappa})} - \frac{\langle S_3, P_{111}^{(\frac{1}{\kappa})} \rangle_{\frac{1}{\kappa}}}{\langle P_{111}^{(\frac{1}{\kappa})}, P_{111}^{(\frac{1}{\kappa})} \rangle_{\frac{1}{\kappa}}} P_{111}^{(\frac{1}{\kappa})}.$$

Macdonald polynomials: defined as the unique family orthogonal *w.r.t.* $\langle, \rangle_{q,t}$ and such that

$$P_\lambda(\mathbb{X}; q, t) = m_\lambda(\mathbb{X}) + \sum_{\mu \leq \lambda} v_{\lambda, \mu}^{q,t} m_\mu(\mathbb{X}).$$

Dual basis: The dual basis of $P_\lambda(q, t)$ is the basis $Q_\lambda(q, t)$ ($Q_\lambda(q, t) = \mathfrak{C}_{\lambda, q, t} P_\lambda(q, t)$).

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Link between Jack and Macdonald polynomials: substituting q by $t^{\frac{1}{\kappa}}$ and then taking the limit $t \rightarrow 1$, we map $P_\lambda(q, t)$ onto $P_\lambda^{\left(\frac{1}{\kappa}\right)}$.

Jack-Selberg integrals

Simple operations and Carlson's theorem yield the following result (that holds for all $\kappa \in \mathbb{C}$) :

$$\langle P_{\lambda}^{\left(\frac{1}{\kappa}\right)} \rangle_{a,b,\kappa,N}^{\#} = \left[\prod_{i=1}^{\ell(\lambda)} \prod_{j=i+1}^{\ell(\lambda)} \frac{\Gamma(\lambda_i - \lambda_j + \kappa(j - i + 1)) \Gamma(\kappa(j - i + 1))}{\Gamma(\lambda_i - \lambda_j + \kappa(j - i)) \Gamma(\kappa(j - i))} \right] \\ \left[\prod_{i=1}^{\ell(\lambda)} \prod_{j=0}^{\lambda_i - 1} \frac{\kappa(N + 1 - i) + j}{\kappa(\ell(\lambda) + 1 - i) + j} \right] \left[\prod_{i=1}^{\ell(\lambda)} \prod_{j=0}^{\lambda_i - 1} \frac{a + \kappa(N - i) + j}{a + b + \kappa(2N - i - 1) + j} \right].$$

Asymptotic behavior of the Jack-Selberg integral:

$$\langle P_{\lambda}^{\left(\frac{1}{\kappa}\right)} \rangle_{a,b,\kappa,N}^{\#} \underset{N \rightarrow \infty}{\sim} N^{|\lambda|}.$$

General method

Let f be a multivariate polynomial, not necessarily symmetric. The algorithm to compute $\langle f \rangle_{a,b,\kappa,N}^\#$ is the following:

- 1 Symmetrize f :

$$\mathfrak{S}f = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_n} \sigma f = \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_n} f(x_{\sigma(1)}, \dots, x_{\sigma(N)})$$

$\langle \mathfrak{S}f \rangle = \langle f \rangle$ since we integrate over an hypercube.

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- 2 Expand $\mathfrak{S}f$ in the $P_\lambda^{(\frac{1}{\kappa})}$ basis;
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If $\mathfrak{S}f$ admits a decomposition in the Jack basis which does not depend on the number of variables, the asymptotics can be exactly computed since the resulting integral is a rational function of N .

Specialization of Macdonald polynomials

Next step: coefficient $\alpha_{\lambda,k}$ of $P_\lambda^{(\frac{1}{\kappa})}$ in p_k ?

We present a simple proof of the value of the coefficient of $P_\lambda(q, t)$ in p_k by investigating the following **specialization of the Macdonald Q polynomials**:

$$Q_\lambda\left(\frac{1-u}{1-t}; q, t\right).$$

The coefficient of $P_\lambda(q, t)$ in p_k is given by

$$(1 - q^k) \frac{\prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (1,1)}} (t^{i-1} - q^{j-1})}{\prod_{s \in \lambda} (1 - q^{a_\lambda(s)+1} t^{l_\lambda(s)})}$$

with the *arm* and *leg* notations:

$$a_\lambda((i, j)) = \lambda_i - j \text{ and } l_\lambda((i, j)) = \lambda'_j - i.$$

The coefficient of $P_\lambda^{(\frac{1}{\kappa})}$ in p_k is equal to

$$\alpha_{\lambda,k} = k \frac{\prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (1,1)}} ((j-1) - \kappa(i-1))}{\prod_{s \in \lambda} (a_\lambda(s) + 1 + l_\lambda(s)\kappa)}.$$

This decomposition **does not depend on N** . Therefore, it is possible to investigate the asymptotic behavior of the power sum - Selberg integral:

$$\langle p_k \rangle_{a,b,\kappa,N}^\# = k \sum_{\lambda \vdash k} \frac{\prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (1,1)}} ((j-1) - \kappa(i-1))}{\prod_{s \in \lambda} (a_\lambda(s) + 1 + l_\lambda(s)\kappa)} \langle P_\lambda^{(\frac{1}{\kappa})} \rangle_{a,b,\kappa,N}^\#.$$

Asymptotics and conjecture

We define $\{p_k\}_{a,b,\kappa} = \lim_{N \rightarrow \infty} \langle p_k \rangle_{a,b,\kappa,N}^\sharp$.

Numerical evidences lead to the following conjectures:

Conjecture 1

If a, b, κ **do not depend** on N , $\{p_k\}_{a,b,\kappa}$ does not depend on a, b or κ and

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Conjecture 2

If a, b, κ are **linear functions** of N , $\frac{1}{N} \langle p_k \rangle_{a,b,\kappa,N}^\#$ still converges and

$$\lim_{N \rightarrow \infty} \frac{1}{N} \langle p_k \rangle_{\kappa(\ell-1)N, 0, \kappa, N}^\# = \frac{\ell}{(1+\ell)^{2k-1}} \sum_{i=0}^{2(k-1)} \binom{k-1}{\lfloor \frac{i}{2} \rfloor} \binom{k-1}{\lfloor \frac{i}{2} \rfloor} \ell^i.$$

Conclusion

- Understand the **simplifications** that give such a simple result;
- Understand the **combinatorial structure** that links these integrals to Dyck paths ($\binom{k-1}{\lceil \frac{i}{2} \rceil} \binom{k-1}{\lfloor \frac{i}{2} \rfloor}$ is the number of symmetric Dyck paths of odd semi-length k with i peaks counted by peaks);
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Merci!