Functional coefficients in solutions of non-commutative differential equations

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Outline

1. Motivation

2. Main Theorem

3. Examples
   - Polylogarithms
   - Counterexample
   - Hyperlogarithms

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- Polyzetas are values of polylogs at 1 :

$$\zeta(\mathbf{s}) = \mathrm{Li}_{\mathbf{s}}(1).$$

Polylogs can be manipulated as shuffles : algebra structure.

- (Non commutative) Differential equation :

Drinfel'd equation $$\frac{d}{dz} T(z) = \left(\frac{1}{z} x_0 + \frac{1}{1-z} x_1 \right) T(z).$$

- Bailey & Borwein & Girgensohn 1994,
- Zagier 1994,
- Flajolet & Salvy 1998,
- Minh & Petitot & Van Der Hoeven 1998,
- Waldshmidt 2000,
- Cartier 2002,
- Racinet 2002,
- Ecalle 2003,
- Cresson & Fischler & Rivoal 2006.

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Examples of relations between Multiple Zeta Values (M. Bigotte) :

$$\zeta(7,5) = -\frac{6117808}{2627625}\zeta(2)^6 - 7\zeta(10,2) + 28\zeta(5)\zeta(7) + 14\zeta(3)\zeta(9)$$

$$\zeta(9,3) = -\frac{93976}{79625}\zeta(2)^6 - \frac{9}{2}\zeta(10,2) + 12\zeta(5)\zeta(7) + 9\zeta(3)\zeta(9)$$

$$\zeta(9,3) = \frac{9}{14}\zeta(7,5) + \frac{388112}{1226225}\zeta(2)^6 - 6\zeta(5)\zeta(7)$$

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Notations

If X is an alphabet and k a field,

- X^* denotes the set of words with letters in X ;
- $k\langle X\rangle$ is the algebra of non commutative polynomials with coefficients in k,
- $k\langle\langle X\rangle\rangle$ the algebra of non commutative series with coefficients in k ;
- If S is a (non commutative) series and w a word, $\langle S|w\rangle$ denotes the coefficient of S on w.

Data

- X an alphabet.
- $(\mathbb{A}, \mathrm{d})$ a commutative differential algebra over the ring k :
    - differential : $\forall\, a, b \in \mathbb{A},\ \mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}(b)$;
    - d is linear over k ;

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  - d is linear over k ;
- We require that $\ker(\mathrm{d}) = k$ (set of constants $= k$).
- Extension of d to $\mathbb{A}\langle\langle X\rangle\rangle$:

$$\forall S \in \mathbb{A}\langle\langle X\rangle\rangle,\ \mathrm{d}(S) = \sum_{w \in X^*} \mathrm{d}(\langle S|w\rangle)w.$$

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- Let $\mathbb{D}$ be a differential subfield of $\mathbb{A}$ (i.e. $\mathrm{d}(\mathbb{D}) \subset \mathbb{D}$).
- M : a homogeneous series of degree 1 :

$$M = \sum_{x \in X} u_x x \in \mathbb{D}_{=1}\langle\langle X\rangle\rangle.$$

Theorem 1 : Linear independence

Suppose that $T \in \mathbb{A}\langle\langle X\rangle\rangle$ is a solution of the differential equation

$$\mathrm{d}T = MT; \langle T|1_{X^*}\rangle = 1.$$

The following conditions are equivalent[1] :

i) The family $(\langle T|w\rangle)_{w\in X^*}$ of coefficients of T is free over $\mathbb{D}$.

ii) The family of coefficients $(\langle T|y\rangle)_{y\in X\cup\{1_{X^*}\}}$ is free over $\mathbb{D}$.

iii) The family $(u_x)_{x\in X}$ is such that, for $f \in \mathbb{D}$ and $\alpha_x \in k$

$$\mathrm{d}(f) = \sum_{x\in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0).$$

iv) The family $(u_x)_{x\in X}$ is free over k and

$$\mathrm{d}(\mathbb{D}) \cap \mathrm{span}_k\left((u_x)_{x\in X}\right) = \{0\}.$$

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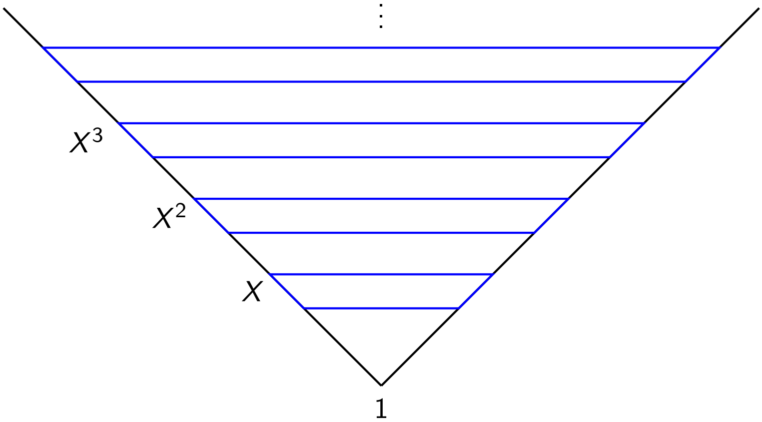
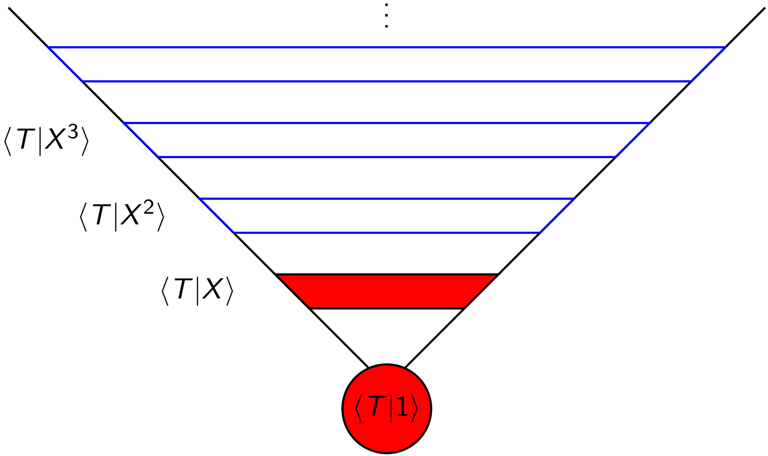
[1] **Independence of hyperlogarithms over function fields via algebraic combinatorics**, M. D., G. H. E. Duchamp, H. N. Minh and A. Solomon, *CAI 2011*, LNCS 6742, pp. 127–139. Springer, Heidelberg (2011)

“Slices“ of the free monoid



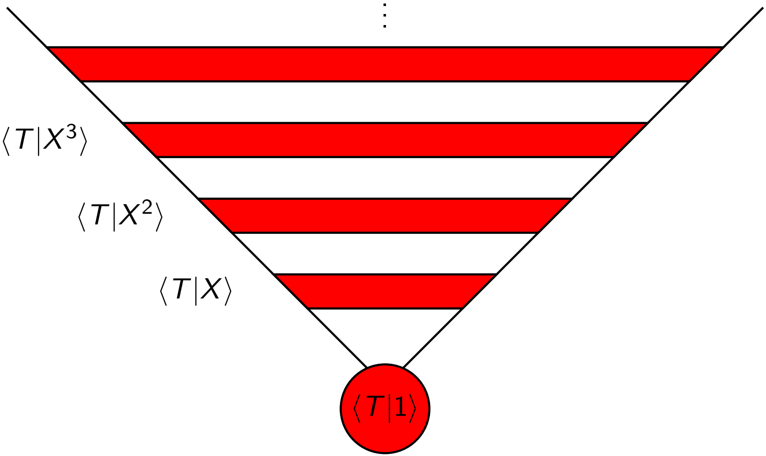
Linear independence of the bottom triangle



Linear independence of the bottom triangle

$\Rightarrow$

Linear independence of the whole triangle



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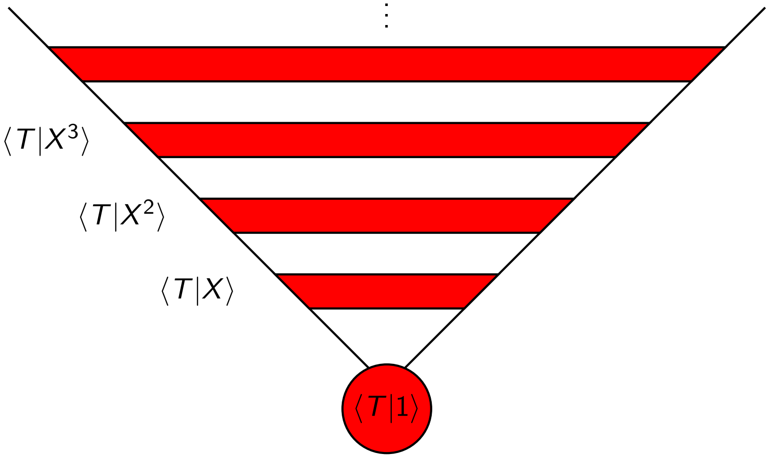

$\Rightarrow$Test of linear relations over a finite number of functions.

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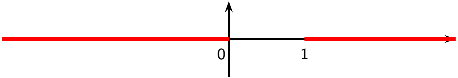
Polylogarithms

$$X = \{x_0, x_1\}.\ \Omega = \mathbb{C}\setminus(]-\infty, 0[\,\cup\,]1, +\infty[).\ u_0(z) = \frac{1}{z},\ u_1(z) = \frac{1}{1-z}.$$



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Definition

$\forall z \in \Omega$,

$$\mathrm{Li}_{x_0^n}(z) = \frac{\ln^n(z)}{n!}.$$

$$\mathrm{Li}_{x_1 w}(z) = \int_0^z \frac{dt}{1-t}\mathrm{Li}_w(t),$$

and, $\forall w \in X^* x_1 X^*$,

$$\mathrm{Li}_{x_0 w}(z) = \int_0^z \frac{dt}{t}\mathrm{Li}_w(t).$$

Remark

Let $w = x_0^{s_1-1}x_1 \ldots x_0^{s_k-1}x_1 \leftrightarrow \mathbf{s} = (s_1, \ldots, s_k)$.
It can be shown that the Taylor expansion of these functions is given by

$$\mathrm{Li}_w(z) = \mathrm{Li}_{\mathbf{s}}(z) = \sum_{n_1>n_2>\cdots>n_k>0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_k^{s_k}}.$$

Application of the theorem

- $\mathbb{A} =$ functions from $\Omega = \mathbb{C} \backslash (]-\infty, 0[\cup]1, +\infty[)$ to $\mathbb{C}$.
- $\mathbb{D} =$ field of functions on Ω (germs of analytic functions).
- $T =$ generating series of polylogs : $T(z) = \sum_{w \in X^*} \mathrm{Li}_w(z) w$.
- $M(z) = \frac{1}{z} x_0 + \frac{1}{1-z} x_1$.
- Differential equation : Drinfel'd equation

$$\frac{d}{dz} T(z) = M(z) T(z).$$

Consequence : Linear independance of polylogs over $\mathbb{D}$.

Let $X = \{x_0, x_1\}$. $u_0(z) = 1$ and $u_1(z) = \frac{1}{z}$.

Encoding integrals : $x_i \to \int_{z_i}^{z} \cdot u_i(s) ds \, x_i = \alpha_{z_i}^{z}(x_i) x_i, \qquad z_0 = 0, \; z_1 = 1.$

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 & 1 \arrow[dl] \arrow[dr] & \\
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Problem : Third condition of the theorem.

iii) The family $(u_x)_{x\in X}$ is such that, for $f \in \mathbb{D}$ and $\alpha_x \in k$

$$\mathrm{d}(f) = \sum_{x\in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0).$$

Hyperlogarithms

Definition (1928, Lappo-Danilevski)

Let $a_0, \ldots a_k \in \mathbb{C}$. Then

$$L(a_{i_n}, \ldots, a_{i_1} | \gamma) = \int_{z_0}^{z} \int_{z_0}^{s_n} \ldots \int_{z_0}^{s_2} \frac{ds_1}{s_1 - a_{i_1}} \ldots \frac{ds_n}{s_n - a_{i_n}}$$

with $\gamma : z_0 \rightsquigarrow z$ a path such that

$$a_{j_i} \notin \gamma \text{ and } s_i \in \gamma, \forall i \in \{1, \ldots, n\}.$$

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- If $z_0 \neq a_{i_1}$, the integral converges.
- Our theorem applies as well to families of inputs of the type

$$u_i(z) = \frac{\lambda_i}{z - a_i}, \lambda_i \in \mathbb{C}^*.$$

Field of germs

Idea : Field of analytic functions with variable domains.

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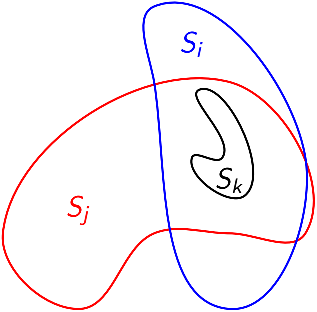
- Ω : connected, simply connected, analytic domain.
- $\mathfrak{B}$: a filter basis of Ω of open connected (non void) subsets of Ω :

$$\forall S_i,\ S_j \in \mathfrak{B},\ \exists S_k \in \mathfrak{B},\ S_k \subset S_i \cap S_j.$$



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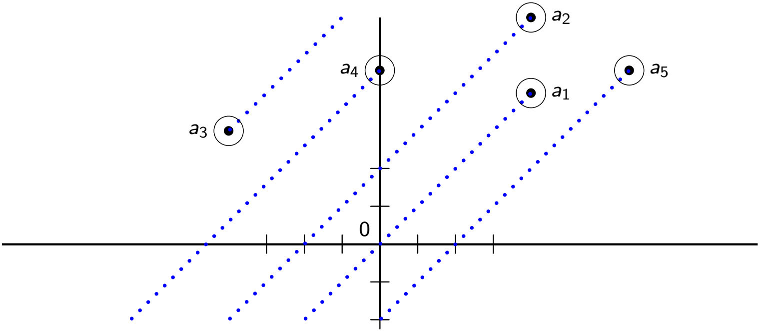
- Compatibility with restrictions : $\forall U, W \in \mathfrak{B}$, $W \subset U$,

$$\mathrm{res}_{WU}(C[U]) \subset C[W] .$$

Example

What is the good domain if the inputs are $u_i(z) = \dfrac{\lambda_i}{z - a_i}$?
It is always possible to cut the complex plane with half rays to form a simply connected domain on which the u_i's are analytic :



Encoding integrals with words

$X = \{x_1, \ldots, x_n\}$.
Definition of the iterated integrals $\alpha_{z_0}^{z}(w)$ for $w \in X^*$ and $z_0, z \in \Omega$:

$$\begin{aligned} \alpha_{z_0}^{z}(1_{X^*}) &= 1; \\ \alpha_{z_0}^{z}(x_i) &= \int_{z_0}^{z} u_i(s)ds,\ x_i \in X; \\ \alpha_{z_0}^{z}(x_i w) &= \int_{z_0}^{z} u_i(s)ds\, \alpha_{z_0}^{s}(w),\ x_i \in X,\ w \in X^*. \end{aligned}$$

Generating series of hyperlogarithms :

$$T(z) := \sum_{w \in X^*} \alpha_{z_0}^{z}(w) w.$$

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General idea : Derivating T term by term, we obtain the following non commutative differential equation :

$$\frac{d}{dz}T(z) = M(z)T(z), \qquad M(z) = \sum_{x_i \in X} u_i(z)x_i.$$

Integrator

Let $M(z) = \sum_{x_i \in X} u_i(z) x_i = \sum_{x_i \in X} \frac{1}{z - a_i} x_i$ and $z_0 \in \mathbb{C}$.
We define the integrator H_{z_0} :

$$H_{z_0} : \begin{array}{c} \mathbb{A}\langle\langle X \rangle\rangle \to \mathbb{A}_{\geq 1}\langle\langle X \rangle\rangle \\ S \mapsto H_{z_0}[S] = \int_{z_0}^{z} M(s) S(s) ds \end{array}$$

Since $\forall S$, $H_{z_0}^n[S] \in \mathbb{A}_{\geq n}\langle\langle X \rangle\rangle$,

$$\langle H_{z_0}^n[S] \,|\, w \rangle \neq 0 \text{ only for } n \leq |w|.$$

Therefore, we can define the sum

$$\sum_{w \in X^*} \sum_{n \geq 0} \langle H_{z_0}^n[S] \,|\, w \rangle w = H_{z_0}^*[S] = \sum_{n \geq 0} H_{z_0}^n[S] \,.$$

(Non commutative) Differential equation

It is clear that

$$H_{z_0}^* = 1 + H_{z_0} H_{z_0}^*$$

Therefore, $\forall\, S \in \mathbb{A}\langle\langle X \rangle\rangle$ such that $dS = 0$ (constant series),

$$\mathrm{d}\left(H_{z_0}^*\,[S]\right) = \mathrm{d}\left(S + H_{z_0}\left(H_{z_0}^*\,[S]\right)\right) = M H_{z_0}^*\,[S]\,,$$

and $H_{z_0}^*\,[S]$ satisfies the (non commutative) differential equation

$$\mathrm{d}P = MP.$$

Since

$$H_{z_0}^*\,[1] = T(z) = \sum_{w \in X^*} \alpha_{z_0}^{z}(w) w$$

we obtain the promised differential equation.

Application of the theorem

- $X = \{x_1, \ldots, x_n\}$.
- $\mathfrak{B}$: filter basis obtained by cutting the complex plane.
- $\mathbb{D} =$ Field of germs of functions on $\mathfrak{B}$ fulfilling condition i) of theorem 1 (*e.g.*, the field of rational functions or the field of functions that are inessential at all the points a_i).
- $T =$ generating series of hyperlogs : $T(z) = \sum_{w \in X^*} \alpha_{z_0}^{z}(w) w$.
- $M(z) = \sum_{i=1}^{n} \frac{1}{z - a_i} x_i, \ \lambda_i \neq 0, \forall i$.

Consequence : Linear independance of hyperlogs over $\mathbb{D}$.

Conclusion and perspectives

Conclusion :

- New proof of known results (without monodromy[2]) ;
- Generalization of these results to wider classes of algebras and functions.

Perspectives : Implementation.

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[2] **Computation of the monodromy of generalized polylogarithms**, Minh & Petitot & Van Der Hoeven, *ISSAC'98*

Thank you for your attention!