Independence of hyperlogarithms over function fields via algebraic combinatorics

Matthieu Deneufchâtel, Gérard H. E. Duchamp,
Vincel Hoang Ngoc Minh, and A. I. Solomon

Laboratoire d'Informatique de Paris Nord,
Université Paris 13

Conference on Algebraic Informatics, June 22, 2011

Outline

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Hyperlogarithms

Definition (1928, Lappo-Danilevski)

Let $a_0, \dots a_k \in \mathbb{C}$. Then

$$L(a_{i_n}, \dots, a_{i_0} | z, z_0) = \int_{z_0}^{z} \int_{z_0}^{s_n} \dots \int_{z_0}^{s_1} \frac{ds_0}{s_0 - a_{i_0}} \dots \frac{ds_n}{s_n - a_{i_n}}.$$

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- If $z_0 \neq a_{i_0}$, the integral converges.

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- If $z_0 \neq a_{i_0}$, the integral converges.
- Largest field on which these functions are independent ?
- If no linear independence, describe the relations.

Polylogarithms

$X = \{x_0, x_1\}$.

Definition

$$\mathrm{Li}_{x_1}(z) = -\ln(1-z) \text{ and } \mathrm{Li}_{x_0}(z) = \ln(z).$$

$$\mathrm{Li}_{x_0 w}(z) = \int_0^z \frac{dt}{t}\mathrm{Li}_w(t); \qquad \mathrm{Li}_{x_1 w}(z) = \int_0^z \frac{dt}{1-t}\mathrm{Li}_w(t).$$

$w = x_0^{s_1-1}x_1 \dots x_0^{s_k-1}x_1 \leftrightarrow \mathbf{s} = (s_1, \dots, s_k)$.

$$\mathrm{Li}_w(z) = \mathrm{Li}_{\mathbf{s}}(z) = \sum_{n_1 > n_2 > \dots > n_k > 0} \frac{z^{n_1}}{n_1^{s_1} \dots n_k^{s_k}}.$$

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$\mathbb{C}$-linear independence [Van der Hoeven, Hoang Ngoc Minh, Petitot, 2000].

Coloured Polylogarithms

Definition

$$\mathrm{Li}_{\mathbf{b},\mathbf{s}}(z) = \sum_{n_1 > n_2 > \cdots > n_k > 0} \frac{b_1^{n_1} \ldots b_k^{s_k}}{n_1^{s_1} \ldots n_k^{s_k}} z^{n_1}$$

with $\mathbf{s} = (s_1, \ldots, s_k)$, $\mathbf{b} = (b_1, \ldots, b_k)$.

Linear independence over $\mathbb{C}\left[z, \frac{1}{z}, \left(\frac{1}{\rho_i^{-1} - z}\right)_{i=0,\ldots,n}\right]$ where the $(\rho_i)_i$'s are the n^{th} roots of 1 [Hoang Ngoc Minh, 2004].

Encoding integrals with words

Let $u_i(z)$, $0 \leq i \leq n$, be n inputs and $X = \{x_1, \ldots, x_n\}$.

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Recursive definition of the iterated integrals $\alpha_{z_0}^{z}(w)$ for $w \in X^*$:

$$\alpha_{z_0}^{z}(1_{X^*}) = 1;$$

$$\alpha_{z_0}^{z}(x_i) = \int_{z_0}^{z} u_i(s)ds,\ x_i \in X;$$

$$\alpha_{z_0}^{z}(x_i w) = \int_{z_0}^{z} u_i(s)ds\, \alpha_{z_0}^{s}(w),\ x_i \in X,\ w \in X^*.$$

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Then if $u_i(z) = \dfrac{1}{z - a_i} \leftrightarrow x_i$,

$$\alpha_{z_0}^{z}(x_{j_0} \ldots x_{j_n}) = L(a_{j_n}, \ldots, a_{j_0} | z, z_0).$$

Generating series of hyperlogarithms :

$$L(z) := \sum_{w \in X^*} \alpha_{z_0}^{z}(w) w.$$

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General idea : Derivating term by term L, we obtain the following non commutative differential equation :

$$\frac{d}{dz}L(z) = M(z)L(z),\ \text{with } M(z) = \sum_{x_i \in X} u_i(z)x_i.$$

Outline

Non commutative polynomials and series

Let R be a (commutative) ring (with unit).

- $R\langle X\rangle$: non commutative polynomials ;
- $R\langle\langle X\rangle\rangle$: non commutative series ;
- For $S \in R\langle\langle X\rangle\rangle$, $\langle S|w\rangle$ is the coefficient of S on the word w :

$$S = \sum_{w \in X^*} \langle S|w\rangle w.$$

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Special case : $R = (\mathfrak{A}, \mathrm{d})$, a commutative differential algebra over k.

- differential : $\forall\, a, b \in \mathfrak{A},\ \mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}(b)$;
- $\mathrm{d}(a + b) = \mathrm{d}(a) + \mathrm{d}(b)$;
- we require (for the following theorems) that $\ker(\mathrm{d}) = k$ (the only constants for the derivation are the elements of the ground field k).

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- $\mathrm{d}(a+b) = \mathrm{d}(a) + \mathrm{d}(b)$;
- $\ker(\mathrm{d}) = k$.

We can extend in a unique way d to $\mathfrak{A}\langle\langle X\rangle\rangle$:

$$\forall S \in \mathfrak{A}\langle\langle X\rangle\rangle,\ \mathrm{d}(S) = \sum_{w\in X^*} \mathrm{d}(\langle S|w\rangle)w.$$

Integrator

Let $M \in \mathfrak{A}_{=1}\langle\langle X\rangle\rangle$ $(M(z) = \sum_{x\in X} u_x(z)x)$ and $z_0 \in \mathbb{C}$.

We define the integrator H_{z_0} :

$$H_{z_0} : \begin{array}{c} \mathfrak{A}\langle\langle X\rangle\rangle \to \mathfrak{A}_{\geq 1}\langle\langle X\rangle\rangle \\ S \mapsto H_{z_0}[S] = \int_{z_0}^{z} M(s)S(s)ds \end{array}$$

with

$$\int_{z_0}^{z} T(s)ds = \sum_{w\in X^*} \left(\int_{z_0}^{z} \langle T|w\rangle ds\right) w, \quad \forall T \in \mathfrak{A}\langle\langle X\rangle\rangle.$$

Iterated integrals

Since $\forall S,\ H_{z_0}^{n}[S] \in \mathfrak{A}_{\geq n}\langle\langle X\rangle\rangle$,

$$\langle H_{z_0}^{n}[S] \,|\, w\rangle \neq 0 \text{ only for } n \leq |w|.$$

Therefore, we can define the sum

$$\sum_{w\in X^*}\sum_{n\geq 0}\langle H_{z_0}^{n}[S] \,|\, w\rangle w = H_{z_0}^{*}[S] = \sum_{n\geq 0} H_{z_0}^{n}[S]\,.$$

Link with hyperlogarithms : With $u_i(z) = \dfrac{1}{z - a_i}$, $\forall i$,

$$H_{z_0}^{*}[1] = L(z) = \sum_{w\in X^*}\alpha_{z_0}^{z}(w)w$$

and

$$\langle H_{z_0}^{*}[1] \,|\, w\rangle = \alpha_{z_0}^{z}(w).$$

(Non commutative) Differential equation

It is clear that

$$H_{z_0}^* = 1 + H_{z_0} H_{z_0}^*$$

Therefore, $\forall\, S \in \mathfrak{A}\langle\langle X\rangle\rangle$ such that $dS = 0$ (constant series),

$$\mathrm{d}\left(H_{z_0}^*\left[S\right]\right) = \mathrm{d}\left(S + H_{z_0}\left(H_{z_0}^*\left[S\right]\right)\right) = MH_{z_0}^*\left[S\right],$$

and $H_{z_0}^*\left[S\right]$ satisfies the (non commutative) differential equation

$$\mathrm{d}T = MT.$$

Outline

1. Introduction
   - Definition
   - Known results
   - Encoding
2. Differential equation
3. Theorems and application to hyperlogarithms

Theorem 1 : Linear independence

Let $\mathfrak{C}$ be a differential subfield of $\mathfrak{A}$ (i.e. $\mathrm{d}(\mathfrak{C}) \subset \mathfrak{C}$).
We suppose that $T \in \mathfrak{A}\langle\langle X \rangle\rangle$ is a solution of the differential equation

$$\mathrm{d}T = MT; \langle T | 1_{X^*} \rangle = 1$$

where M is a homogeneous series of degree 1 : $M = \sum_{x \in X} u_x x \in \mathfrak{C}_{\geq 1}\langle\langle X \rangle\rangle$.

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The following conditions are equivalent :

i) The family $(\langle T | w \rangle)_{w \in X^*}$ of coefficients of T is free over $\mathfrak{C}$.

ii) The family of coefficients $(\langle T | y \rangle)_{y \in X \cup \{1_{X^*}\}}$ is free over $\mathfrak{C}$.

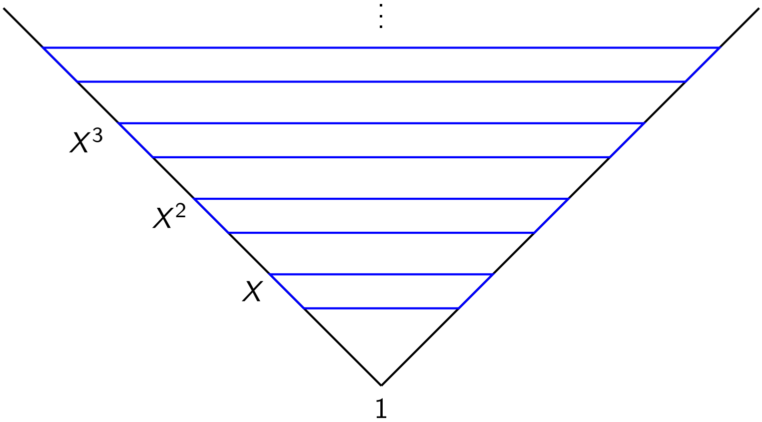
iii) The family $(u_x)_{x \in X}$ is such that, for $f \in \mathfrak{C}$ and $\alpha_x \in k$

$$\mathrm{d}(f) = \sum_{x \in X} \alpha_x u_x \Longrightarrow (\forall x \in X)(\alpha_x = 0).$$

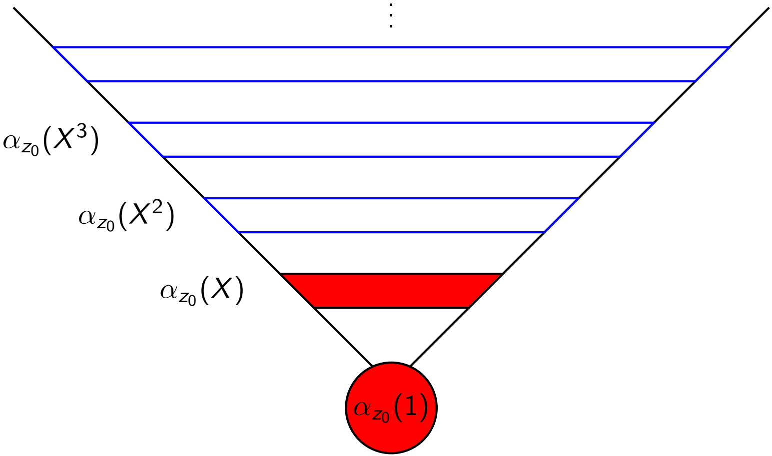
iv) The family $(u_x)_{x \in X}$ is free over k and

$$\mathrm{d}(\mathfrak{C}) \cap \mathrm{span}_k \left((u_x)_{x \in X} \right) = \{0\}.$$

“Slices“ of the free monoid



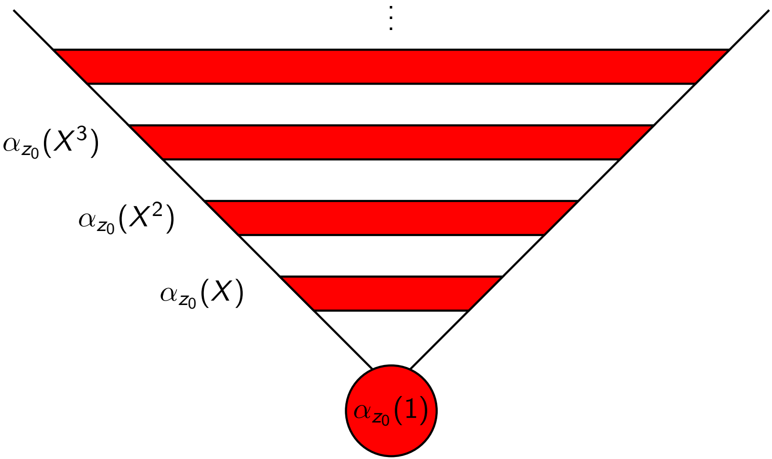
Linear independence of the "first triangle"



Linear independence of the "first triangle"

$\Rightarrow$

Linear independence of the whole triangle



Field of germs

Idea : Field of analytic functions with variable domains.

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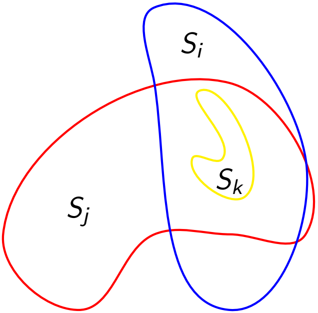
- V : connected, simply connected, analytic variety : $\mathbb{C}\setminus\{a_1, \ldots, a_n\}$;
- $\mathfrak{B}$: a filter basis of V of open connected subsets of V :

$$(\mathbb{C}\setminus(a_j)_{j\subset\mathscr{J}})_{\mathscr{J}\in\mathscr{I}};$$



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- $\mathfrak{B}$: $(\mathbb{C} \setminus (a_j)_{j \subset \mathscr{J}})_{i \in \mathscr{I}}$;
- A correspondence C such that

  $\forall U \in \mathfrak{B},\ C[U]$ is a subring of $\mathscr{C}^{\omega}(U, \mathbb{C})$, satisfying :

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- Inverse : if $f\in C[U]\setminus\{0\}$, $\exists W\in\mathfrak{B}$ such that $W\subset U-\mathscr{O}_f$ and f^{-1} (defined on W) is in $C[W]$.
- Compatibility with restrictions : $\forall U,W\in\mathfrak{B},\ W\subset U$,

$$\mathrm{rest}_{WU}(C[U])\subset C[W].$$

Theorem 2 : Application to hyperlogarithms

Let

- $M = \sum_{i=1}^{n} \frac{\lambda_i}{z - a_i} x_i$ be a multiplier on X with all $\lambda_i \neq 0$,
- S be any regular solution of

$$\frac{d}{dz} S(z) = M(z) S(z).$$

Then, let C be a differential field of functions defined on V which does not contain linear combinations of logarithms on any domain but which contains z and the constants (as, for example, the rational functions). If $U \in \mathfrak{B}$ and $P \in C[U]\langle X \rangle$, one has

$$\langle S | P \rangle = 0 \Longrightarrow P = 0.$$

Linear independence on $C[U]$.

Application : Hyperlogarithms

Corollary of the previous theorem :

R denotes the ring of functions that can be analytically extended to some $V \cup U_{a_1} \cup U_{a_2} \cup \cdots \cup U_{a_n}$ (U_{a_i} open neighborhood of a_i) and have non-essential singularities at these points.

Then the set of hyperlogarithms $(\langle S|w\rangle)_{w\in X^*}$ is a a set of linearly independent functions over R.

Conclusion and perspectives

Conclusion :

- New and simpler proof of known results (without monodromy) ;
- Generalization of these results to a wider class of algebras.

Perspectives :

- Find the maximal field on which the linear independence is preserved ;
- Write a program that
  - tests the linear (in)dependence of functions over a given differential ring ;
  - gives the relations between the coefficients if necessary.

Thank you for your attention!