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Why Delannoy's numbers?

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Algorithms Project

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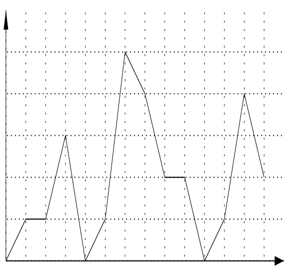
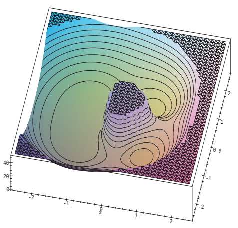
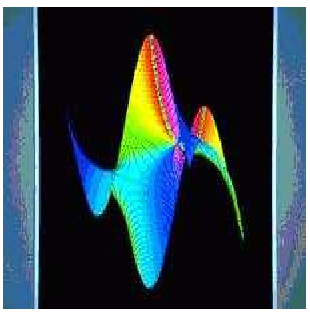






Part I. Lattice Path Problems: Delannoy numbers

(Mathematics from the XIXst century...)

Famous integer sequences.
Famous lattice paths.

- Leonardo Fibonacci (∼1170–∼1250) [*Liber abbaci* 1202]
- Blaise Pascal (1623–1662)
- Jacob Bernoulli (1654–1705)
- James Stirling (1692–1770)
- Leonhard Euler (1707–1783)
- Eugène Catalan (1814–1894)[1838]
- Ernst Schröder (1841–1902) [*4 comb. Probleme* 1870]
- Walther von Dyck (1856–1934)
- Jan Łukasiewicz (1878–1956)
- Eric Temple Bell (1883–1960)
- Theodore Motzkin (1908-1970) [1948]

Catalan numbers: Netto 1958 [Segner/Euler 1759]
Schröder numbers: Comtet 1970 [Hipparchus of Rhodes 150BC]
Motzkin numbers: Donaghey & Shapiro 1977
Delannoy numbers: ???

Delannoy numbers

1	21	221	1561	8361	36365	134245	433905	1256465	3317445	8097453
1	19	181	1159	5641	22363	75517	224143	598417	1462563	3317445
1	17	145	833	3649	13073	40081	108545	265729	598417	1256465
1	15	113	575	2241	7183	19825	48639	108545	224143	433905
1	13	85	377	1289	3653	8989	19825	40081	75517	134245
1	11	61	231	681	1683	3653	7183	13073	22363	36365
1	9	41	129	321	681	1289	2241	3649	5641	8361
1	7	25	63	129	231	377	575	833	1159	1561
1	5	13	25	41	61	85	113	145	181	221
1	3	5	7	9	11	13	15	17	19	21
1	1	1	1	1	1	1	1	1	1	1

central Delannoy numbers $D_{n,k}$

construction rule: a+b+c=d

Some formulae...

$$D(z) := \sum_{n\geq 0} D_{n,n} z^n = [x^0]\, \frac{1}{1 - z\,(x + 1/x + 1)} = \frac{1}{\sqrt{1 - 6z + z^2}}$$

$$= 1 + 3\,z + 13\,z^2 + 63\,z^3 + 321\,z^4 + 1683\,z^5 + 8989\,z^6 + 48639\,z^7 + O(z^8)\,.$$

$$D_{n,n} = \frac{(3+2\sqrt{2})^n}{\sqrt{\pi}\sqrt{3\sqrt{2}-4}} \left(\frac{n^{-1/2}}{2} + \frac{23\,n^{-3/2}}{32(8+3\sqrt{2})} + \frac{2401\,n^{-5/2}}{2048(113+72\sqrt{2})} + O(n^{-7/2}) \right)$$

$$\approx 5.82842709^n \left(.57268163\,n^{-1/2} + .06724283\,n^{-3/2} + .00625063\,n^{-5/2} + \ldots\right).$$

$$D_{n,k} = \sum_{i=0}^{n} \binom{n}{i} \binom{k}{i} 2^i$$

Fortuitous and useless: $D_{n,n} = P_n(3)$ (Legendre polynomial).

So, why Delannoy's numbers?

Because...

Henri Auguste Delannoy (1833-1915).
Polytechnician, Military intendant.

Italy (1859: Solférino battle, 24 June)
Africa (1866-1869), directed a hospital in Alger (during a typhus epidemic) → Officier de la Légion d'Honneur

Delannoy & Mathematics

Collaboration with Lucas from 1880 to 1891, published some *Récréations mathématiques*

Using a chessboard to solve arithmetical problems (1886)

$\frac{n-k+1}{n+1}\binom{n}{n+k}$ for the number of Dyck paths length n altitude k Desiré André (1887) ballot numbers:
$T_{x,y} = \binom{x}{x+y} - \binom{x-1}{x+y} = \frac{y-x+1}{y+1}\binom{x}{x+y}$

Length of the game (1888): Dyck walk in the strip $[-n, n]$

$$\frac{(-1)^{m-n}}{n}\sum_{k=1}^{n}(-1)^{k-1}\sin\left(\frac{(2k-1)\pi}{2n}\right)\cos^{m-1}\left(\frac{(2k-1)\pi}{2n}\right)$$

$$= \quad \frac{n}{2^{m-1}}\sum_{k=0}^{q/n}(-1)^k\frac{2k+1}{\frac{m+n}{2}+kn}\binom{m-1}{q-kn}.$$

Rouché/Feller/Knuth

How to use a chessboard to solve various probability theory problems (1889)
potpourri of 7 (ballot-like or ruin-like) problems (partially) solved by Laplace, Huyghens, Ampère, Rouché, Bertrand, André, ...

Delannoy's "chessboard": triangular, rectangular, pentagonal, hexagonal. (Lucas' *Number theory*, 1891).

Various problems about the game (1890): $\frac{1}{\sqrt{2\pi}}\sqrt{2n}$

Formulae related the binomial coefficients (1890)

$$\sum_{k=0}^{p}(p-2k)^2\binom{p}{k} = p2^p$$

On the geometrical trees and their use in the theory of chemicals compounds. (1894)
(bi) centered hydrocarbons with n atoms (Cayley)

How to use a chessboard to solve some probability theory problems (1895)
17 applications of his theory (gives the array of Delannoy numbers)

On a question of probabilities studied by d'Alembert (1895)

On the probability of simultaneous events (1898)
contesting the "third Laplace principle"
$P(A) \cap P(B) = P(A)P(B)$ for two independent events.

"L'Intermédiaire des Mathématiciens" (1894-1908)
70 contributions: combinatorics (enumeration and applications to probabilistic problems), elementary number theory (representations of integers as sum of some powers, Fermat-like problems), and questions/answers related to Lucas' books (recreative mathematics, e.g., four color problem).

Other Delannoy's works

1896-1915: président from the Société des Sciences Naturelles et archéologiques de la Creuse.

1897 to 1914: 29 archaeological/historitical articles

"On the signification of word *ieuru*"
"One more word about *ieuru*"
"A riot in Guéret in 1705"
"Aubusson's tapestries"
"A bigamist in Guéret"
"Grapevines in the Creuse"
studies "Criminal trials in the Marche. The case ... "
studies on abbeys and some "Critical list of the abbots from ... "
"An impotence trial in the 18th century"
...

When he died, at the age of 81, about a dozen of other articles were still in progress.

Part II. A "new way" to tackle these lattice paths problems: The kernel method

(Mathematics from the XXIst century...)

Walks on $\mathbb{N}$

Aim : Find the generating function of a "walk on the integers"

$$F(z,u) = \sum_{n\geq 0, k\geq 0} f_{n,k} u^k z^n = \sum_{n\geq 0} f_n(u) z^n = \sum_{k\geq 0} F_k(z) u^k$$

n encodes the length $\quad$ k encodes the final altitude

- $f_{n,k}$: # paths of length n going from 0 to k.
- $F(z,1)$ counts all the paths .

Weight of a path = product of the weight of its jumps

Model with multiplicity: weight $\in \mathbb{N}$.

Probabilistic model : weight $\in [0,1]$.

names	jumps	$P(u)$ Laurent p.
Dyck	$\{-1,+1\}$	$\frac{1}{u}+u$
Motzkin	$\{-1,0,+1\}$	$\frac{1}{u}+1+u$
Motzkin bicolores	$\{-1,0,0,+1\}$	$\frac{1}{u}+2+u$
Coin tossing	$\{-1,+1\}$	$\frac{p}{u}+qu$
General case	$\{-c,\ldots,+d\}$	$p_{-c}u^{-c}+\cdots+p_d u^d$

Dyck paths

Ballot problem : $f_{n,k} :=$ # possible ballots with n electors, during which candidate A is always ahead B, knowing that A wins finally by k votes.

Walk on $\mathbb{N}$ with jumps $\{-1, +1\}$ $\qquad P(u) = u^{-1} + u$

$$F(z,u) = \sum_{n,k \geq 0} f_n(u) z^n$$

$$f_{n+1}(u) = \{u^{\geq 0}\} P(u) f_n(u)$$

$$f_{n+1}(u) = P(u) f_n(u) - \frac{1}{u}[u^{-1}]\frac{1}{u} f_n(u)$$

$$F(z,u) = 1 + z\left(P(u)F(z,u) - \frac{1}{u}F_0(z)\right)$$

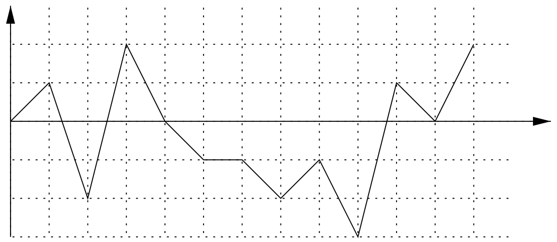
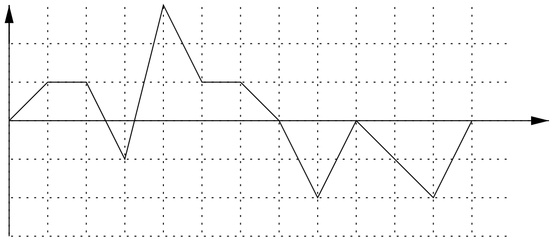
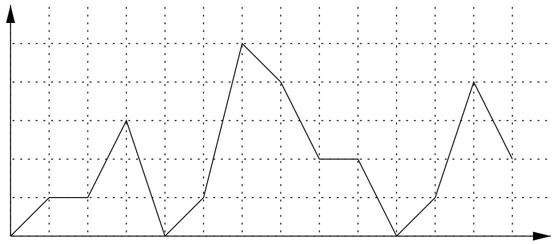
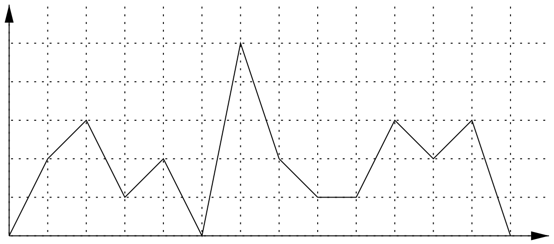
$$\big(1 - zP(u)\big)F(z,u) = 1 - \frac{z}{u}F_0(z)$$

Kernel method [folklore: TAOCP 2.2.1 exercice 4]

cancelling $1 - zP(u)$: $u - z(1+u^2) = 0$

$\to$ two roots $u_1(z) \sim z$ and $u_2(z) \sim 1/z$ at $z = 0$

$$0 = 1 - \frac{z}{u_1(z)} F_0(z) \quad \Rightarrow \quad F(z,u) = \frac{u - u_1(z)}{u - z(1+u^2)}$$

walks	ending anywhere	ending in 0
$\subset \mathbb{Z}$	 walk $\quad W_n = P(1)^n$ $F_{\mathbb{Z}}(z,1) = \frac{1}{1-zP(1)}$	 bridge $\quad B_n \sim \beta_0 \frac{P(\tau)^n}{\sqrt{2\pi n}}$ $F_{\mathbb{Z},0}(z) = z \sum_{i=1}^{c} \frac{u_i'(z)}{u_i(z)}$
$\subset \mathbb{N}$	 meander $M_n \sim \mu_0 \frac{P(1)^n}{\sqrt{\pi n}}$ or $\mu_0^- \frac{P(\tau)^n}{2\sqrt{\pi n^3}}$ or $\mu_0^+ P(1)^n$ $F_{\mathbb{N}}(z,1) = \frac{1}{1-zP(1)} \prod_{i=1}^{c} (1 - u_i(z))$	 excursion $E_n \sim \epsilon_0 \frac{P(\tau)^n}{2\sqrt{\pi n^3}}$ $F_{\mathbb{N}}(z,0) = \frac{(-1)^{c+1}}{p_{-c}} \prod_{i=1}^{c} u_i(z)$

$$1 - zP(u_i) = 0$$

Algebraicity (walks on $\mathbb{Z}$)

Proposition 1.

$$F_0(z) = z \sum_{u_i(z)} \frac{u_i'(z)}{u_i(z)} = -z \sum_{v_i(z)} \frac{v_i'(z)}{v_i(z)}$$

where $u_i(z)$ and $v_i(z)$ are the roots (finite and infinite at $z = 0$) of $1 - zP(u) = 0$.

Proof:

$$[u^k]F(z,u) = [u^k]\frac{1}{1 - zP(u)} = \frac{1}{2i\pi}\int_{\gamma(z)} \frac{1}{1 - zP(u)}\frac{du}{u^{k+1}}$$

where γ is simple path in a crown of CV of $F(z,u)$ ($|z| < \frac{1}{2P(1)}$ and $\alpha < |u| < \beta$).

$$F_k(z) = \sum_{u_i \text{ pole}} \mathrm{Res}\left(\frac{F}{u^{k+1}}, u_i\right) = \sum_{u_i \text{ pole}} \frac{1}{-zu_i^{k+1}P'(u_i)}$$

$$= z \sum_{u_i \text{ pole}} \frac{u_i'(z)}{u_i^{k+1}(z)} \qquad (\text{as } P(u_i) = \frac{1}{z}).$$

For $k < c$: shorten the contour, residues$= u_i$.
For $k > -d$: enlarge the contour, residues$= v_i$.

Algebraicity (walks on $\mathbb{N}$)

Theorem 1. *GF for excursions/paths on $\mathbb{N}$ is*

$$F_0(z) = \frac{(-1)^{c+1}}{z} \prod_{u_i} u_i(z)$$

$$F(z,1) = \frac{\prod_{u_i}(1-u_i)}{1-zP(1)}$$

(product of the finite roots u_i of the kernel $1-zP(u)$).

Proof: $P(u) = u^{-c} + \cdots + u^d$

$$f_{n+1}(u) = P(u) f_n(u) - \{u^{<0}\} P(u) f_n(u)$$

$$(1-zP(u)) F(z,u) = \text{ LC of } F_j(z)$$

$1 - zP(u) = 0 \implies c$ finite roots u_i

LC= polynomial (!) in u of degree c

factorization $\implies F(z,u) = \prod (u-u_i)/(u^c - zu^c P(u))$

$\square$

For free, as corollary: Spitzer–Andersen formulae

$$F_{\mathbb{Z},0}(z) = 1 + z\frac{d}{dz}\log(F_{\mathbb{N},0}(z)),$$

$$F_{\mathbb{N},0}(z) = \exp\left(\int_0^z \frac{F_{\mathbb{Z},0}(t)-1}{t} dt\right).$$

Height of a walk on $\mathbb{N}$

Theorem 2. *bivariate generating fucntion $F^{[h]}(z,u)$ for walks of height $\leq h$ is a linear combination of $c+d$ quotients of determinants.*

$$F_0^{[h]}(z) = \frac{\det M_1^{[h]}}{\det M^{[h]}}\,.$$

Proof: $$f_{n+1}^{[h]} = P f_n^{[h]} - \{u^{<0}\} P f_n^{[h]} - \{u^{>h}\} P f_n^{[h]}$$

$$(1 - zP(u))\, F^{[h]}(z,u) = 1 - z\left(\sum_{k=0}^{c-1} r_k(u) F_k^{[h]}(z) + \sum_{k=0}^{d-1} \tilde{r}_k^{[h]}(u) F_{h-k}^{[h]}(z)\right).$$

$c+d$ unknowns, the kernel $1-zP$ has $c+d$ roots.

$$\begin{pmatrix}
r_0(u_1) & r_1(u_1) & \dots & r_{c-1}(u_1) & \tilde{r}_{d-1}^{[h]}(u_1) & \dots & \tilde{r}_0^{[h]}(u_1)\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
r_0(u_c) & r_1(u_c) & \dots & r_{c-1}(u_c) & \tilde{r}_{d-1}^{[h]}(u_c) & \dots & \tilde{r}_0^{[h]}(u_c)\\
r_0(u_{c+1}) & r_1(u_{c+1}) & \dots & r_{c-1}(u_{c+1}) & \tilde{r}_{d-1}^{[h]}(u_{c+1}) & \dots & \tilde{r}_0^{[h]}(u_{c+1})\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
r_0(u_{c+d}) & r_1(u_{c+d}) & \dots & r_{c-1}(u_{c+d}) & \tilde{r}_{d-1}^{[h]}(u_{c+d}) & \dots & \tilde{r}_0^{[h]}(u_{c+d})
\end{pmatrix}
\begin{pmatrix}
F_0^{[h]}\\ \vdots \\ F_{c-1}^{[h]} \\ F_{h-d+1}^{[h]} \\ \vdots \\ F_h^{[h]}
\end{pmatrix}
=
\begin{pmatrix}
\frac{1}{z}\\ \vdots \\ \frac{1}{z} \\ \frac{1}{z} \\ \vdots \\ \frac{1}{z}
\end{pmatrix}$$

complexity $= (c+d)^3 = O(1)$ independent of h ($\neq$ Markov chain). > [de Bruijn–Knuth–Rice, 1972]

Directed walks in dimension 2

chess knight jumps encoded by

$P(u,v) := uv^2 + uv^{-2} + u^2v + u^2v^{-1}$

$$F(z,u,v) = 1 + z\big(P(u,v)F(z,u,v)$$
$$-uv^{-1}F_1(z,u) - (u^2v^{-1} + uv^{-2})F_0(z,u)\big)$$

"number of paths going from $(0,0)$ to $(k,0)$" :

$$a_k = [u^k v^0]F(1,u,v).$$

$$F(1,u,0) = \frac{1+2u+\Delta}{4u} - \frac{\sqrt{2+2\Delta-8u^2+4u\Delta}}{4u}$$

$\Delta := \sqrt{1-4u+4u^2-4u^4}$ $\qquad a_k \asymp (1+\sqrt{3})^k \approx 2.73^k \quad \square$

context free grammars approach

Labelle & Yeh, 1990, "Generalized Dyck languages"

less convenient than the kernel method

Kernel Method: a success story

$\Longrightarrow$ Enumeration (even if $|\text{alphabet}| = \infty$)

C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, D. Gouyou-Beauchamps “Generating functions for generating trees” 1998

$\rightarrow$ application to the ECO method of Pinzani & al., $(k) \rightarrow (2) \cdots (k)(k+1)$

M. Bousquet-Mélou & M. Petkovšek “Linear recurrences . . . ” 1998

M. Bousquet-Mélou & G. Schaeffer “Walks on the slit plane” 2001

C. Banderier & D. Merlini “Lattice paths with an infinite set of jumps” 2002

also possible : walks under rational slope, higher Markovian orders.

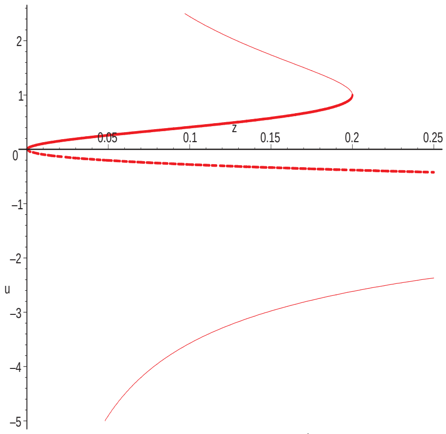
$\Longrightarrow$ Asymptotics

G. Fayolle & R. Iasnogorodski & V. Malyshev “Random walks in the quarter-plane” 1999 $\leftarrow$ Riemannian geometry

C. Banderier & Ph. Flajolet “Analytic combinatorics of lattice paths” 2001

C. Banderier “Limit laws for parameters of walks with unbounded jumps” 2002

Asymptotics : $\mathbb{Z}$ and $\mathbb{N}$ excursions



Theorem 3. *(complete asymptotic expansions)*

$$E_{\mathbb{Z},n} \sim \beta_0 \frac{P(\tau)^n}{\sqrt{2\pi n}} \left(1 + \frac{b_1}{n} + \frac{b_2}{n^2} + \cdots \right)$$

$$E_{\mathbb{N},n} \sim \epsilon_0 \frac{P(\tau)^n}{\sqrt{2\pi n^3}} \left(1 + \frac{e_1}{n} + \frac{e_2}{n^2} + \cdots \right),$$

Constants are algebraic : $P'(\tau) = 0$, $\rho := \frac{1}{P(\tau)}$,
$\beta_0 = \frac{1}{\tau}\sqrt{\frac{P(\tau)}{P''(\tau)}}$, $\epsilon_0 = \frac{(-1)^{c-1}}{p_{-c}} \sqrt{\frac{P(\tau)^3}{2P''(\tau)}} \prod_{j=2}^{c} u_j(\rho)$.

Proof (saddle point method and singularity analysis)

- $E_{\mathbb{N}}(z)$ same radius of CV as $E_{\mathbb{Z}}(z)$
- $E_{\mathbb{N}}(z) = \prod_{i=1}^{c} u_i(z)$
- u_i are conjugated : $U_i(t) = U_1(w^{j-1}t)$
- $\exists$! saddle $\tau > 0$ such that $P'(\tau) = 0$, $\rho := P(\tau)$
- $u_1(z) - \tau = d_1\sqrt{\rho - z} + d_2(\rho - z) + \ldots$
- $[z^n]\sqrt{1-z} = \frac{-1}{2\sqrt{\pi n^3}}(1 + O(\frac{1}{n}))$

Theorem 4 (Asymptotics of $\mathbb{N}$-paths).

drift $\delta = P'(1)$

$$f_n(1) \sim \alpha(1) \frac{P(1)^n}{\sqrt{\pi P''(\tau)} n^{1/2}} + \dots \quad \delta = 0,$$

$$f_n(1) \sim \frac{P^2(\tau)\,\alpha(\tau)}{P(\tau) - P(1)} \frac{P(\tau)^n}{\sqrt{\pi P''(\tau)} n^{3/2}} + \dots \quad \delta < 0,$$

$$f_n(1) \sim (1 - \tau_1)\,\alpha(1) P(1)^n + \frac{P^2(\tau)\,\alpha(\tau)}{P(\tau) - P(1)} \frac{P(\tau)^n}{\sqrt{\pi P''(\tau)} n^{3/2}} + \dots \quad \delta > 0.$$

Constants all computable (*τ is the real saddle point of $P(u)$, $\rho = \frac{1}{P(\tau)}$*)

Proof:

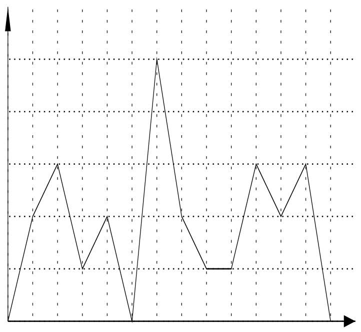
$$F(z,1) = \frac{1}{1 - zP(1)} \prod_{i=1}^{c} (1 - u_i(z)) \, .$$

$\delta = 0 : F(z,1) \sim \frac{1-u_1(z)}{1-zP(1)} \prod_{i=2}^{c} (1 - u_i(\frac{1}{P(1)})),$

$\delta < 0 : F(z,1) \sim \frac{1}{1-\rho P(1)} (1 - (\tau - d_1\sqrt{\rho - z}))\ \alpha(\tau) \sim -\sqrt{\rho - z} \frac{d_1 P(\tau)}{P(1)-P(\tau)}\ \alpha(\tau)\, ,$

$\delta > 0 : F(z,1) \sim \frac{1}{1-zP(1)} (1 - \tau_1)\, \alpha(1)\, .$ □

Returns to 0 for $\mathbb{N}$-excursions



Excursion = sequence of arches

Theorem 5. *The probability that an $\mathbb{N}$-excursion of length n has j returns to 0 is*

$$\frac{1}{E(\rho)^2}(j+1)\left(1-\frac{1}{E(\rho)}\right)^j + O\left(\frac{1}{n}\right).$$

$\sim$ *sum of 2 i.i.d. random variables (geometric of parameter $1-E(\rho)^{-1}$).*

Proof: number of excursions with j arches :

$$e_{nj} = (1-\frac{1}{E(z)})^j \sim [z^n]\frac{1}{E(\rho)} - \frac{j(1-\frac{1}{E(\rho)})^{j-1}\sqrt{r-z}}{E(\rho)^2}$$

$p_j = e_{nj}/e_n \to$ sum of two geometric R.V. $\square$

Theorem 6 (Final altitude of $\mathbb{N}$-paths).

$\delta < 0$, *converges to a discrete limit law:*

$$\lim_{n\to\infty} \Pr(X_n = k) = [u^k] \frac{(1-\tau)^2}{(u-\tau)^2} \prod_{\ell \geq 2} \frac{1 - v_\ell(\rho)}{u - v_\ell(\rho)}.$$

$\delta = 0$, *convergences to a Rayleigh distribution (density $xe^{-x^2/2}$) :*

$$\lim_{n\to\infty} \Pr\left(\frac{X_n}{\sqrt{\frac{P''(1)}{P(1)}\, n}} \leq x \right) = 1 - e^{-x^2/2},$$

$\delta > 0$, *converges to a Gaussian limit law :*

$$\lim_{n\to\infty} \Pr\left(\frac{X_n - \mu n}{\sigma\sqrt{n}} \leq x \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

where $\mu = \frac{P'(1)}{P(1)}$ *and* $\sigma^2 = \frac{P''(1)}{P(1)} + \frac{P'(1)}{P(1)} - \left(\frac{P'(1)}{P(1)}\right)^2.$

Conclusion

Remember Delannoy & Think kernel method !

N.B.: a conference related to Delannoy numbers, lattice path combinatorics & probability theory ...
will be organized in 2003 in France (in Guéret, where Delannoy lived).