**Università di Firenze**
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# Coupon collector problems and random walks on graphs.

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# Supper Conjecture

(submitted during a supper at the meeting Analysis of Algorithms in 1999, at Barcelona)

- $m$ guests around a table
- probability $p$ : pour into my glass some Brunello di Montalcino
- give the bottle randomly to my right or left neighbor, etc.

$T(p) := \#$ bottle moves before everyone has got some wine.

Conjecture

$$pE[T(p)] \to 0 \times \infty = mH_m \text{ when } p \to 0 \text{ ?}$$

$(H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m})$

What about $E[T(p)]$ if people give the bottle preferentially to their friends?

# The Classical Coupon Collector Problem

time needed to collect $m$ images
=covering the complete graph (equiprobability)

$$11^*2\{1,2\}^*3\{1,2,3\}^* \dots \{1,2,\dots,m-1\}m$$

probability generating function:

$$f(x_1,\dots,x_m) = \frac{x_m}{m}\prod_{j=1}^{m-1}\frac{x_j/m}{1-(x_1/m+\cdots+x_j/m)}$$

How many doubles, triples...?

$$\sum_{i=1}^{\infty} E_i z^i = m! \sum_{i=1}^{m} f(1,\dots,1,x_i,1,\dots,1)$$

$E_i$ := expected number of different coupons in $i$-double

$$\sum_{i=1}^{\infty} E_i z^i = t-1+\frac{m!}{\prod_{j=2}^{m}(j-z)} \quad \Box$$

Hyperharmonic numbers [Foata–Zeilberger, 2001]

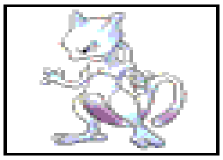
# Trivial Pursuit for dummies

random walk on a graph

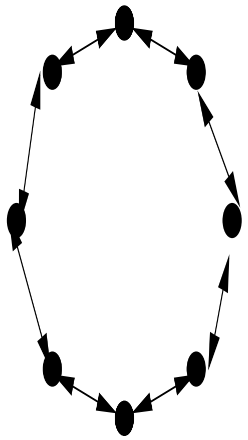
cover time:= first time that every vertex has been hit by the walk

marking time:= first time that every vertex is marked (when the walk reaches vertex $v_i$, with probability $p_i$ the vertex is *marked*)

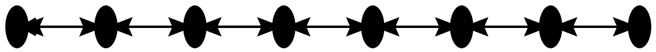
cover time = coupon collector problem (Pokemon cards!)

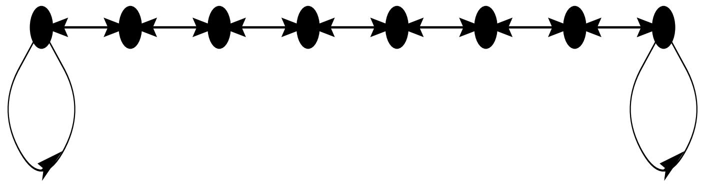
marking time = two levels of randomness (Trivial Pursuit!)



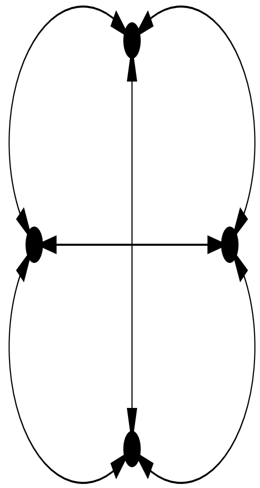
Cyclic graph (m=8)



Linear graph with reflexion at buttoms (m=8)



Linear graph with loops at buttoms (m=8)



Complete graph (m=4)

Can you guess the marking time of these graphs ?

# Notations

Graph

- $G$ = directed connected graph with vertices $v_1, \ldots, v_m$
- Transition matrix $A$ ( $a_{ij}$ is the probability of a transition from $v_i$ to $v_j$).
- Stationary distribution: $\pi_1, \ldots, \pi_m$.
- Probability to mark the vertex $v_i$ whenever it is visited: $p_i$ $(i = 1, \ldots, m)$, $q_i := 1 - p_i$.

Operators

$[z^2]\,(1 + 4z^2 + z^3) = 4.$

For $m = 4$ and $\alpha = \{1, 2, 4\}$

$\zeta_\alpha\,(3u_1 + u_1u_3 + u_3^3 + u_4) = u_3^3$ and

$\sigma_\alpha\,(3u_1 + u_1u_2 + u_3^3 + u_4) = 3q_1 + q_1q_2 + 1 + q_4.$

# Marking time on any graph

**Theorem 1 (General formula for average marking time).** *The average time for marking all the vertices of an arbitrary graph $G$ is given by*

$$E[T(\{p_1,\ldots,p_m\})] = \sum_{\alpha\neq\emptyset}(-1)^{|\alpha|+1}\sigma_\alpha \sum_{i=1}^{m} u_1(\mathrm{Id}-AU)^{-1}_{1i},$$

*where $A$ is the transition matrix, $U = diag(u_1,u_2,\ldots,u_m)$*

Proof ${A^n}_{i,j}$:= probability of moving from vertex $v_i$ to $v_j$ in $n$ steps.
$[u_1^{k_1}\cdots u_m^{k_m}](AU)^n_{i,j}$:= idem + vertex $v_t$ is visited $k_t$ times

$$F(z,u_1,\dots,u_m) = u_1 \sum_{i=1}^{m} \sum_{k=0}^{\infty} z^k (AU)^k_{1,i} = u_1 \sum_{i=1}^{m} (\mathrm{Id} - zAU)^{-1}_{1,i}.$$

marking all the vertices at least once $\Longrightarrow$ substitution $u_i^n \leftarrow 1 - q_i^n$

$\Longrightarrow$ difference operator $\Delta_i f(u_i) := f(1) - f(q_i)$.

Generating function of the walks that marked *all* the vertices:

$$F^{\mathrm{all}}(z) = \Delta_1 \Delta_2 \dots \Delta_m F = \sum_{\alpha \subseteq \{1,\dots,m\}} (-1)^{|\alpha|} \sigma_\alpha F(z,u_1,\dots,u_m),$$

$$E[T(\{p_1,\dots,p_m\})] = \frac{\partial}{\partial z}_{|z=1} (1-z) F^{\mathrm{all}}(z).$$

A change of variable $1 - z = t$ and a local development in $t = 0$ gives

$$\frac{\partial}{\partial z}_{|z=1} \sigma_\alpha \left( \frac{(1-z)u_1}{\mathrm{Id} - zAU} \right)_{ij} = -\sigma_\alpha \left(\mathrm{Id} - AU\right)^{-1}_{ij} \quad \square$$

Remark. Setting the $p_i$'s to 1 $\Rightarrow$ coupon collector problem.

# Marking time on the complete graph

**Theorem 2.** *Let $\widetilde{G}$ be a graph with transition probabilities $\widetilde{a}_{i,j} := \pi_j$. The expected time $E[\widetilde{T}]$ for marking the whole graph $\widetilde{G}$ is $1/p$ times the time needed for visiting all the vertices*

$$E[\widetilde{T}] = \frac{1}{p} E[\textit{coupon collector on } \widetilde{G}],$$

*and an inclusion-exclusion formula holds*

$$E[\widetilde{T}] = \frac{1}{p} \sum_{\alpha \neq \emptyset} \frac{(-1)^{m-|\alpha|}}{-\zeta_\alpha(\pi_1 + \pi_2 + \cdots + \pi_m)},$$

*where the $\zeta_\alpha$'s replace some $\pi_i$ by 0.*

Proof Consider the $\pi_i's$, the $q_i$'s, and the $p_i$'s as formal variables, then

$$\mathrm{Prob}(\widetilde{T} \leq n) = \{p_1^{>0}\} \ldots \{p_m^{>0}\}(\pi_1 p_1 + \pi_1 q_1 + \cdots + \pi_m p_m + \pi_m q_m)^n,$$

where $\{p_1^{>0}\}p_1 p_2^n + 1 + p_2\}$ stands for the sum of monomials in $p_1$ of positive exponent. Thus, we have

$$\mathrm{Prob}(\widetilde{T} \leq n) = \sum_{\alpha \subseteq \{1,\ldots,m\}} (-1)^{|\alpha|}(\zeta_\alpha(\pi_1 p_1 + \pi_1 q_1 + \cdots + \pi_m p_m + \pi_m q_m))^n,$$

$$\sum \mathrm{Prob}(\widetilde{T} \leq n) z^n = \sum_{\alpha \subseteq \{1,\ldots,m\}} \frac{(-1)^{|\alpha|}}{1 - z(q + p\zeta_\alpha(\pi_1 + \pi_2 + \cdots + \pi_m))}.$$

Multiplying by $1 - z$ gives $\sum \mathrm{Prob}(\widetilde{T} = n) z^n$
and differentiating in $z = 1$ gives the expected time

$$E[\widetilde{T}] = \sum_{\alpha \neq \emptyset} (-1)^{|\alpha|} \frac{1}{p(-1 + \zeta_\alpha(\pi_1 + \pi_2 + \cdots + \pi_m))} \qquad \square$$

| | graph $G$ | graph $\widetilde{G}$ |
|---|---|---|
| transition matrix | $A = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$ | $\widetilde{A} = \begin{pmatrix} 3/8 & 3/8 & 1/4 \\ 3/8 & 3/8 & 1/4 \\ 3/8 & 3/8 & 1/4 \end{pmatrix}$ |
| stationary distribution | $(3/8,\ 3/8,\ 1/4)$ | $(3/8,\ 3/8,\ 1/4)$ |
| | $U = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix}$ | $U = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix}$ |
| generating matrix of the walks | $(\mathrm{Id} - zAU)^{-1}$ | $(\mathrm{Id} - z\widetilde{A}U)^{-1}$ |
| expected covering time | $95/12 \simeq 7.91$ | $29/5 \simeq 5.80$ |
| expected marking time (closed form) | $\dfrac{435 + 494p + 187p^2 + 24p^3}{75p + 60p^2 + 9p^3}$ | $29/5p^{-1}$ |
| expected marking time (asymptotics) | $E[T(p)] = 29/5p^{-1} + O(1)$ | $E[\widetilde{T}(p)] = 29/5p^{-1}$ |

**Proposition 1.** *The marking time $T$ can be approximated by $\widetilde{T}$:*

$$E[\widetilde{T}] - E[T] = O(1), \qquad (p \to 0).$$

Proof In order to prove that

$$\sum_{\alpha \neq \emptyset} (-1)^{|\alpha|+1} \sigma_\alpha \sum_{i=1}^{m} u_1 (\mathrm{Id} - \widetilde{A}U)^{-1}_{1i} - \sum_{\alpha \neq \emptyset} (-1)^{|\alpha|+1} \sigma_\alpha \sum_{i=1}^{m} u_1 (\mathrm{Id} - AU)^{-1}_{1i} = O(1)$$

it is sufficient to check that

$$[p^{-1}]\sigma_\alpha \left(\mathrm{Id} - \widetilde{A}U\right)^{-1}_{ij} - [p^{-1}]\sigma_\alpha \left(\mathrm{Id} - AU\right)^{-1}_{ij} = O(1).$$

- $\alpha = \{1, \ldots, m\}$ eigenprojection
- $\alpha \neq \{1, \ldots, m\}$ perturbation of two noninvertible matrices

- $\alpha = \{1, \ldots, m\}$ $A = P + R$ (so $P^n = P = \widetilde{A}$ and $PR = RP = 0$)

$(1 - zA)^{-1} = \sum z^n (P + R)^n = \frac{\widetilde{A}}{1-z} + (\mathrm{Id} - zR)^{-1}$
with $(\mathrm{Id} - zR)^{-1}$ regular in $z = 1$ (by Perron–Frobenius theory) so
$(1 - zA)^{-1} = \frac{\widetilde{A}}{1-z} + O(1)$.

- $\alpha \neq \{1, \ldots, m\}$ perturbation of two noninvertible matrices

$$(\mathrm{Id} - A + \epsilon B)^{-1} = \frac{\widetilde{A}}{-\epsilon \lambda'(0)} + O(1) = (\mathrm{Id} - \widetilde{A} + \epsilon \widetilde{B})^{-1}$$

where $B$ is $A$ with its $i$-th column set to 0 if $i \in \alpha$ (so $A - \epsilon B = \sigma_\alpha A U$),
and where $\lambda(\epsilon)$ is the perturbation of the eigenvalue 1, so
$\lambda'(0) = \sum_{i \in \alpha} \pi_i$. □

# Probabilistic proof

One wants

$$E[T] = E[\widetilde{T}] + O(1).$$

Note

$$P(T \leq n) = \sum_{\lambda=(n_1,\dots,n_m)} p_\lambda c_\lambda,$$

by definition

$$E[T] = \sum_{n\geq 0} (1 - \mathrm{Prob}(T \leq n)).$$

limit theorem on Markov Chains, $p_\lambda$ follows a multidimensional Gaussian law $g(\lambda)$, thus for a "central interval" $I$ and a "central box" $B$

$$E[T] = \sum_{n\in I} \left(1 - \sum_{\lambda\in B} g(\lambda) c_\lambda\right) + o(p^{-1}).$$

$I$ and $B$ only rely on the stationary distribution $\Longrightarrow E[T] = E[\widetilde{T}] + o(p^{-1}) = E[\widetilde{T}] + O(1)$, as $E[\widetilde{T}]$ and $E[T]$ are *rational fractions* in $p$. $\square$

# Integral form

- previous formulae are impracticable as they comprise $2^m$ summands
- "integral formula" computes $E[T]$ to any precision in linear time

**Theorem 3 (First order asymptotics. Integral form).** *For any graph $G$ with a stationary distribution $(\pi_1, \ldots, \pi_m)$, the expected marking time is*

$$E[T] = \frac{K}{p} + O(1), \qquad \textit{where} \quad K = \int_0^\infty (1 - \prod_{j=1}^{m} (1 - \exp(-\pi_j x)))\, dx.$$

*[Flajolet–Gardy–Thimonier: "Birthday paradox and coupon collector..." 1992]*

Proof set $p' := -\ln(1-p)$

$$E[T] = \sum_{n \in I} \left( 1 - \sum_{\lambda \in B} g(\lambda) \prod_{j=1}^{m} 1 - \exp(-p' n_j \pi_j) \right) + O(1).$$

$$= \sum_{n \geq 0} (1 - \prod_{j=1}^{m} (1 - \exp(-p' n \pi_j))) + O(1) + o(p^{-1}).$$

Euler-Maclaurin formula with $f(x) := 1 - \prod_{j=1}^{m} (1 - \exp(-p' x \pi_j)$:

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(x) dx = \frac{f(0) + f(\infty)}{2} + \int_0^{\infty} (x - \lfloor x \rfloor - 1/2) f'(x) dx = O(1).$$

so

$$E[T] = \frac{1}{p'} \int_0^{\infty} (1 - \prod_{j=1}^{m} (1 - \exp(-\pi_j x)))\, dx + O(1) \text{ when } p' \to 0.$$

Since $p' = -\ln(1-p) = p + o(p)$, one has $E[T] = K/p + O(1)$. $\square$

# Balanced graphs

outdegree of each vertex = its indegree, each outgoing edge is equally likely.

**Lemma 1 (Stationary distribution for balanced graphs).**
*For balanced graphs, one has* $\pi_i = \frac{N_i}{\sum_1^m N_i}$
*($N_i$ is the number of incoming edges of vertex $v_i$).*

Proof $N_{ij}$:=#of edges from $v_i$ to $v_j$, $N_{*j}$:=# incoming edges to $v_j$, $N_{i*}$:=# outgoing edges from $v_i$. The left eigenvector (for eigenvalue 1) satisfies

$$(\pi_1, \pi_2, \ldots, \pi_m) \begin{pmatrix} N_{11}/N_{1*} & \ldots & N_{1m}/N_{1*} \\ \vdots & \vdots & \vdots \\ N_{m1}/N_{m*} & \ldots & N_{mm}/N_{m*} \end{pmatrix} = (\pi_1, \pi_2, \ldots, \pi_m);$$

hence $\sum_i \pi_i \frac{N_{ij}}{N_{i*}} = \pi_j$. This equation is indeed satisfied by $\pi_i := \frac{N_{*i}}{\sum_1^m N_{*i}}$ when $N_{i*} = N_{*i}$. $\square$

**Theorem 4 (Balanced graphs).** *For balanced graphs with $N := \sum N_i$ edges, one has*

$$E[T] = \frac{N}{p} \sum_{\alpha \neq \emptyset} \frac{(-1)^{m-|\alpha|}}{-\zeta_\alpha(N_1 + N_2 + \cdots + N_m)} + O(1),$$

*where the substitutions $\zeta_\alpha$ operate on the $\{N_j\}$.*

## Regular graphs

graphs whose *all* vertices have the same number of incoming and outgoing edges

**Theorem 5 (Regular graphs).** *When all the $\pi_i$'s are equal (in particular if $G$ is a regular graph and outgoing edges are chosen uniformly at random), one has $E[T] = \frac{mH_m}{p} + O(1)$.*

Proof

$$E[\widetilde{T}] = \sum_{i=1}^{m} (-1)^i \binom{m}{i} \frac{1}{p(-1 + \frac{m-i}{m})} = \frac{mH_m}{p}.$$

(forward finite difference operators and Euler's transform) $\square$

| size | graph | stationary distribution | expected cover time | expected marking time |
|---|---|---|---|---|
| $m=1$ | $K_1$ | $(1)$ | 1 | 1 |
| $m=2$ | $K_2$ | $(1/2,\ 1/2)$ | 2 | 3 |
| $m=3$ | $K_3$ | $(1/3,\ 1/3,\ 1/3)$ | 4 | $3H_3{=}11/2$ |
| $m=3$ | $L_3$ | $(1/4,\ 1/2,\ 1/4)$ | 5 | $19/3$ |
| $m=4$ | $L_4$ | $(1/6,\ 1/3,\ 1/3,\ 1/6)$ | 10 | $99/10$ |
| $m=4$ | $K_4$ | $(1/4,\ 1/4,\ 1/4,\ 1/4)$ | $13/2$ | $4H_4{=}25/3$ |
| $m=4$ | $C_4$ | $(1/4,\ 1/4,\ 1/4,\ 1/4)$ | 7 | $4H_4{=}25/3$ |
| $m=4$ | $Y_4$ | $(1/5,\ 2/5,\ 1/5,\ 1/5)$ | 10 | $110/10$ |
| $m=4$ | $T_4$ | $(1/8,\ 3/8,\ 1/4,\ 1/4)$ | $163/15$ | $369/35$ |
| $m=4$ | $Q_4$ | $(3/10,\ 1/5,\ 1/5,\ 3/10)$ | $69/10$ | $62/7$ |

# Conclusion

We got, for any graph
- closed forms formulae for the marking time,
- asymptotics,
- practicable formula.

Easy behavior: *Complete graph with loops*
coupon collector time: $mH_m$
marking time: $mH_m p^{-1}$

General behavior: *any graph*
marking time: related to the stationary distribution.

Finally the Supper Conjecture holds for all *regular graphs*.

Open problem:

Estimate *your* knowledges in sports, history, ... and then say how long will a typical Trivial Pursuit game last for you ?

(NP-hard, $\ln n$-approximable)