Algorithms Seminars
Rocquencourt 1999

Algebraicity and asymptotics of unidimensional random walks

Part I
Generating functions & generating trees

Part II
Other combinatorial applications (CF, 2D)

Part III
Asymptotics (by SA or SPM)

FPSAC/SFCA
Barcelona 1999

On generating functions on generating trees.

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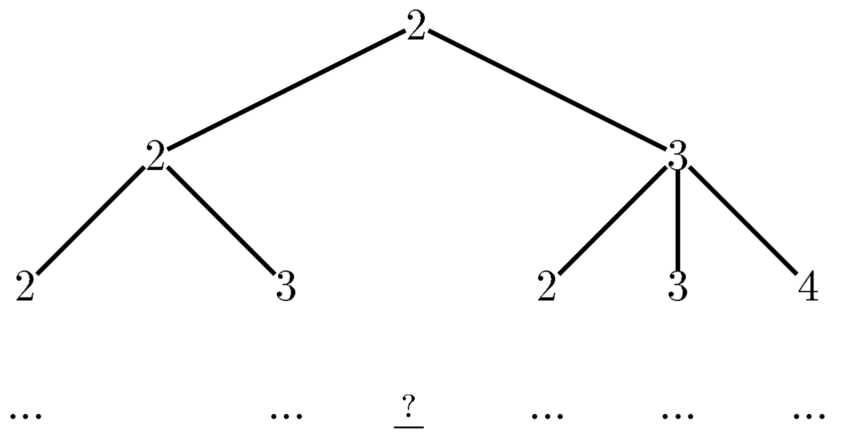
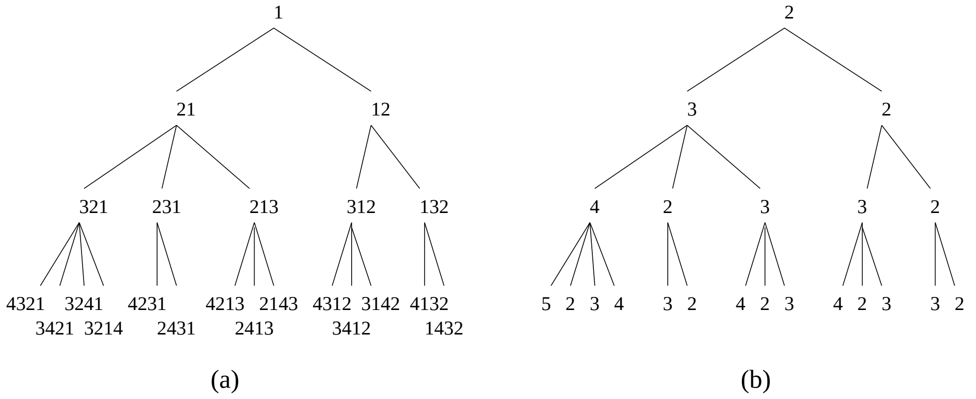


$$\stackrel{?}{=} \frac{1-\sqrt{1-4z}}{2z}$$

Enumerating Combinatorial Objects

J. West, S. Dulucq, O. Guibert: generating tree of 123-avoiding permutations



(a) nodes labelled by the permutations

(b) nodes labelled by the numbers of children

Basic fact: ECO-method of R. Pinzani & al.
$\Longleftrightarrow$ rewriting rules (ecosystem)
$(k) \rightarrow e_1(k) \dots e_k(k)$

Usefulness: Random generation
! non trivial weights for the branches :
e.g. $\Pr(213) = \frac{1}{6}$ but $\Pr(132) = \frac{1}{4}$
$\Longrightarrow$ need enumeration

manual

123 avoiding permutations/generating tree

manual

grammar/generating tree

Generating Trees

	# of nodes	Σ labels
2	1 nodes	2
2 3	2 nodes	5
2 3 2 3 4	5 nodes	14
2 3 2 3 4 2 3 2 3 4 2 3 4 5	14 nodes	42
$\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$ $\vdots$	f_n nodes	s_n

Generating trees: node with label $k \Rightarrow k$ children

$\exists$ associated rewriting rules e.g. (above): axiom $= (2)$ and $(k) \rightarrow (2)(3)\ldots(k+1)$.

Remark: For generating trees $f_{n+1} = s_n$

Question: How many nodes at depth n ?

Generating functions

Aim: Make explicit the generating function of the generating tree

$$F(z) = \sum_{n\geq 0} f_n z^n$$

Equivalent points of view

words from an infinite "regular" grammar
≈ traces in an infinite automaton
≈ iterated power of an infinite matrix
≈ generating trees
≈ random walks on the integer

Generating function of a "walk on the integer":

$$F(z,u) = \sum_{n\geq 0, k\geq 0} f_{n,k} u^k z^n = \sum_{n\geq 0} f_n(u) z^n$$

$n \leftrightarrow$ length $\qquad k \leftrightarrow$ height

•$f_{n,k}$: # walks with length n which start from the initial point and end at level k
= # nodes with label k at depth n in the tree.

•$F(z,1)$ counts all the walks.

•$F(z,0)$ counts all the excursions (final height$=0$).

manual

ecosystems

Problem

What kind of ecosystems gives a

$$\begin{cases} rational \\ algebraic \\ transcendental \end{cases}$$

ordinary generating function ?

Three interesting families

Ecosystem of general shape:

$$(k) \rightarrow e_1(k) \ldots e_k(k)$$

- Rational ecosystems:
  strong regularity conditions
  $(k) \rightarrow (2)^{k-1}(k+1)$ concentrated in $O(1)$
- Algebraic ecosystems:
  factorial form with slight modifications
  $(k) \rightarrow (2)(3) \ldots (k)(k+1)$ diffused $[1..\Theta(k)]$
- Transcendental ecosystems:
  -coefficients grow too fast
  $(k) \rightarrow (k)^{k-1}(k+1)$ concentrated in $O(k)$
  -rules are too "irregular"
  $\begin{cases} (k) \rightarrow (2)^{k-2}(2)(k+1) & \text{when } k \text{ is a power of 2} \\ (k) \rightarrow (2)^{k-2}(3)(k+1) & \text{otherwise} \end{cases}$

Rational Systems I

$$[(s_0),(k) \rightarrow e_1(k)\dots e_k(k)]$$

Theorem 1. *If* $e_1(k)+\cdots+e_k(k)=\alpha k+\beta$, *then*

$$F(z) \;=\; \frac{1+(s_0-\alpha)z}{1-\alpha z-\beta z^2}.$$

Proof:

$$f_{n+2}=s_{n+1}=(\alpha k_1+\beta)+\cdots+(\alpha k_{f_n}+\beta)$$

$$=\alpha s_n+\beta f_n=\alpha f_{n+1}+\beta f_n$$

$\square$

Examples: Fibonacci sequence.
$[(2),\ \{(k)\rightarrow(2)^{k-1}(k+1)\}]$ gives
$e_1(k)+\cdots+e_k(k)=3k-1$ thus
$F(z)=\frac{1-z}{1-3z+z^2}=1+2z+5z^2+\cdots$

(odd entries in the Fibonacci sequence).

$[(2),\ \{(k)\rightarrow(1)^{k-1}(2k)\}]$
$[(2),\ \{(k)\rightarrow(2)^{k-2}(3-(k \bmod 2))(k+(k \bmod 2))\}]$
$[(2),\ \{(k)\rightarrow(2)^{k-2}(3-[k \text{ prime}])(k+[k \text{ prime}])\}]$
Same $F(z)$ but different generating trees.

Rational Ecosystems II

Theorem 2. *Let* $\pi_{i,k} :=$ # *of "one-step transitions from* i *to* k*". If all the series* $\sum_{i\geq 1} \pi_{i,k}\, t^i$ *are rational then* $F(z)$ *is rational.*

Proof: let $F_k(z)$ be the GF of #nodes with label k.

$$F(z) = \sum_{k\geq 1} F_k(z) = \sum_{i=1}^{C} F_i(z) \left[\sum_{k\geq 1} P_{i,k}(z)\right].$$

$\sum_{k\geq 1} P_{i,k}(z)$ rational as a sum of polynomials in z which can made explicit.

Remains to check that the $F_i(z)$ are rational.

But we also have a system (for $k \leq C$)

$$F_k = z \sum_{i=1}^{C} F_i(z) \left[\sum_{j\geq 1} P_{i,j}(z)\pi_{j,k}\right].$$

$\sum_{j\geq 1} P_{i,j}(z)\pi_{j,k}$ is rational (Hadamard product of two rational series).

$\Rightarrow$ linear system with rational coefficients

$\Rightarrow$ F_k rational $\Rightarrow$ F rational. $\square$

Rational Ecosystems III

Consequences of previous theorems:

The prime ecosystem paradox

$\exists$ "prime" generating tree (set of label = set of prime numbers) with rational GF. Proof: Vaughan theorem allows an affine weight. $\square$

Mixing two rules

axiom = (2) and

$$\begin{cases} (k) \rightarrow (2)^{k-2}(3)(k+1) \text{ when } k \text{ is a multiple of } 2 \\ (k) \rightarrow (2)^{k-2}(2)(k+1) \text{ otherwise} \end{cases}$$

$\Rightarrow$ rational GF.

Compatibility with arbitrary forward jumps

$[(3),(k) \rightarrow (666)^{k-1}(1999k+2000)]$

$\Rightarrow$ rational GF.

The prime ecosystem paradox

Letter from Golbach to Euler (1742):
"all odd $n \geq 9$ is sum of 3 primes."
Vinogradov 1937 "OK for large n."
Chen 1973 "for large even n, $n = p + qr$"
Vaughan 1975 "all even n sum ≤ 26 primes"
Deshouillers 1976 "all n sum of ≤ 26 primes"
Riesel Vaughan 1983 "all even n, sum of ≤ 18 "
Ramaré 1995 "all even n sum of ≤ 6 primes"

Idea: (assuming Golbach conjecture for simplicity)

$$2 \to (3)(5)$$

$$\vdots$$

$$p_n \to (p_{n+1})(q_1)(q_2)(q_3)(2)^{k-4}$$

where $q_1 + q_2 + q_3 = 6p_n - p_{n+1} - 2(p_n - 4)$
(> 0 thanks to Bertrand property) so
$\sigma(k) = 6k \Longrightarrow$ rational GF
and set of labels = set of primes

Algebraic Ecosystems

Getting the functional equation.

ex.: $[(0),(k) \to (0)(1)\dots(k+1)]$

$$u^k \to u^0+\dots+u^{k-1}+u^k+u^{k+1} = \frac{1-u^k}{1-u}+u^k+u^{k+1}$$

The basic recurrence is

$$f_{n+1}(u) = \frac{f_n(1)-f_n(u)}{1-u} + (1+u)f_n(u)$$

Multiplying by z^{n+1} and summing (for $n \geq 0$) yields

$$F(z,u) = f_0(u)+z\frac{F(z,1)-F(z,u)}{1-u}+z(1+u)F(z,u)$$

The functional equation is

$$(1-z(1+u)+z\frac{1}{1-u})F(z,u) = 1+z\frac{F(z,1)}{1-u}$$

The Kernel Method

Theorem 3. *Factorial ecosystem ⇒ algebraic GF. More precisely*

$$F(z,1) = -\frac{1}{z}\prod_{i=0}^{c}(1-u_i(z)),$$

Solving the functional equation: the kernel method

$(k) \to (0)(1)\cdots(k-3)(k-1)(k+4)(k+15)$ thus $P(u) := -u^{-2} + u^4 + u^{15}$

$$F(z,u) = 1 + z\left(\frac{F(z,1)-F(z,u)}{1-u} + P(u)F(z,u) - \{u^{<0}\}\sum_{n\geq 0} z^n L[f_n](u)\right).$$

$$F(z,u)\left(1+\frac{z}{1-u} - zP(u)\right) = 1 + \frac{z}{1-u}F(z,1) - z\sum_{j=0}^{c-1} c_j(u)\partial_u^j F(z,0).$$

$$F(z,u)K(z,u) = \text{lin. comb. of } (c+1) \text{ unknown functions.}$$

Cancelling the kernel gives $c+1$ convergent branches.
linear system ⇒ the $\partial_u^j F(z,0)$ are known ⇒ F is algebraic. □

Transcendental Systems I

$$(k) \to e_1(k) \ldots e_k(k)$$

Proposition 1. *Consider a system such that:*

1. *finitely many e_i's are bounded;*
2. *for all k, there exists a forward jump from k (i.e., $e_i(k) > k$ for some i).*

Then the (ordinary) generating function $F(z)$ has radius of convergence zero.

Proof: coefficients of F grow like a factorial. $\square$

Examples: Arrangements

$[(2), (k) \to (k)(k+1)^{k-1}]$ generates the sequence $1, 2, 5, 16, 65, 326 \ldots$. Triangular array $f_{n,k}$ given by the arrangement numbers $k!\binom{n}{k}$ of EGF: $e^z/(1-z)$.

Stirling polynomials

$[2, (k) \to (k)^{k-1}(k+1)]$ Stirling numbers of the second kind $\left\{ n \atop k \right\}$ (the number of ways one can group n objects into k nonempty subsets).

$$\widetilde{F}(z, u) = \sum_{n \geq 0} \left(\sum_{k=1}^{n} \left\{ n \atop k \right\} u^k \right) \frac{z^n}{n!} = \exp(u(\exp(z) - 1)).$$

$u = 1 \to$ Bell

Puzzle

The previous examples are transcendental with radius of convergence 0.

¿ Are there ecosystems with a trancendental GF *and* a positive radius of convergence ?

YES ☐

NO ☐

Tick the right box and win 100 pta.

Transcendental Systems II

An example of transcendence due to irregularity but with radius of convergence > 0:

The Fredholm case axiom $= (3)$ and

$$\begin{cases} (k) \to (2)^{k-2}(2)(k+1) \text{ when } k \text{ is a power of } 2 \\ (k) \to (2)^{k-2}(3)(k+1) \text{ otherwise} \end{cases}$$

involves the "Fredholm series"

$$h(z) := \sum_{m \geq 1} z^{2^m}$$

$h(z)$ has the unit circle as a natural boundary because

$$h(z) = z^2 + h(z^2) \Rightarrow \forall n \quad h(\exp(\frac{2i\pi}{2n})) = \infty.$$

$$F(z) = \frac{z(1-z)^2 h(z)}{(1-2z)(1-z)^2 h(z) - z^4}.$$

radius of convergence $\approx 0.360102 \neq 0$. $\square$

Non Holonomic Transcendental Systems

F is D-finite (holonomic) $\Longleftrightarrow$ F satisfies a linear differential equation with polynomial coefficients in z.

Nonholonomic example:
Bessel histories.
axiom $= (2)$, $(k) \to (k-1)(k)^{k-2}(k+1)$
Classical combinatorial theory of continued fractions gives

$$F(z,0) = \frac{1}{1 - z - z^2 B(z)}$$

where $B(z)$ is the GF of "Bessel numbers".
From [FlSc90], we know that

$$1 - z^2 B(-z) \sim z \frac{J_{1/z-1}(2)}{J_{1/z}(2)},$$

(J_ν: Bessel J-function of order ν)
known asymptotic non compatible with holonomy [WiZe85]
$\Rightarrow F(z,u)$ and $F(z,1)$ are not holonomic. $\square$

A catalog of ecosystems of combinatorial interest

Ecosystem	Name	Id.	Generating Function
	Rational GF's		(ogf)
$(1),\ (k) \to (k)^{k-1}((k \bmod 2)+1)$	Fibonacci	M0692	$\frac{1}{1-z-z^2}$
$(2),\ (k) \to (2)^{k-1}(k+1)$	odd Fibonacci	M1439	$\frac{1-z}{1-3z+z^2}$
$(3),\ (k) \to (2)^{k-1}(k+1)$	even Fibonacci	M2741	$\frac{1}{1-3z+z^2}$
	Algebraic GF's		
$(1),\ (k) \to (1)\cdots(k-1)(k+1)$	Motzkin numbers	M1184	$\frac{1-z-\sqrt{1-2z-3z^2}}{2z}$
$(2),\ (k) \to (2)\cdots(k)(k+1)$	Catalan numbers	M1459	$\frac{1-2z-\sqrt{1-4z}}{2z}$
$(t{+}1),\ (k) \to (t{+}1)\cdots(k)(k+1)^t$	t=2: Schröder numbers	M2898	$\frac{1-2tz-z+z^t-\sqrt{(1+z-tz)^2-4z}}{2tz}$
$(3),\ (k) \to (3)\cdots(k+2)$	Ternary trees	M2926	equation: $F=1+zF^3$
$(t),\ (k) \to (t)\cdots(k+t-1)$	t-ary trees		equation: $F=1+zF^t$
	Transcendental GF's		(egf)
$(2),\ (k) \to (k-1)^{k-1}(k+1)$	Involutions	M1221	$e^{z+\frac{1}{2}z^2}$
$(2),\ (k) \to (k-1)^{k-2}(k)(k+1)$	Switchboard problem	M1461	$e^{2z+\frac{1}{2}z^2}$
$(2),\ (k) \to (k)(k+1)^{k-1}$	Arrangements	M1497	$e^z/(1-z)$
$(2),\ (k) \to (k-1)^{k-2}(k+1)^2$	Bicolored involutions	M1648	e^{2z+z^2}
$(2),\ (k) \to (k+1)^k$	Factorial numbers	M1675	$1/(1-z)$
$(2),\ (k) \to (k+1)^{k-1}(k+2)$	Increasing subsequences	M1795	$e^{z/(1-z)}/(1-z)$
	Nonholonomic GF's		
$(1),\ (k) \to (k)^{k-1}(k+1)$	Bell numbers	M1484	e^{e^z-1}
$(t),\ (k) \to (k)^{k-t}(k+1)^t$	labelled balls in t-col. urns	M1662	$e^{t(e^z-1)}$
$(2),\ (k) \to (k-1)(k)^{k-2}(k+1)$	Bessel numbers	M1462	—

Then, what ?

- obtain automatically the EGF
  (bijective approach [Corteel])
- random generation
- solve more general functional eqns
  (recurrence approach [Petkovšek])
- asymptotics
  $f_n \sim \frac{A\rho^{-n}}{\sqrt{\pi n}}$ (algebraic case)

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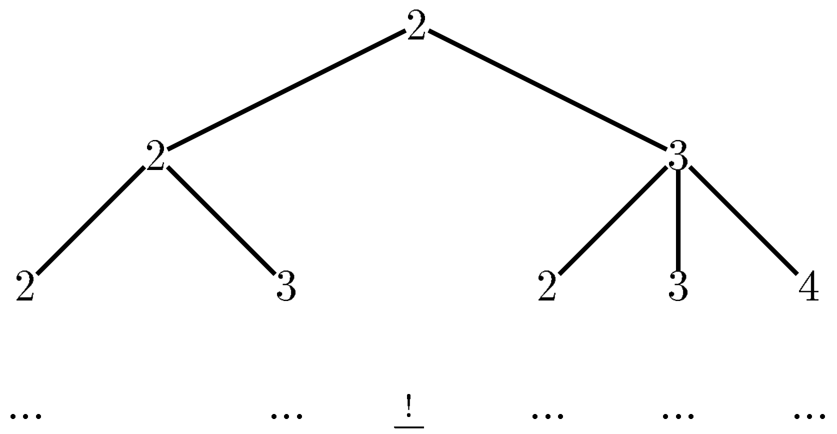


$$\overset{!}{=}$$

$$\frac{1-\sqrt{1-4z}}{2z}$$

Algebraicity of Walks GF

Theorem 4. *GF for excursions/paths of walks with [all backward steps and other] steps in a finite set is algebraic and one has*

$$F(z,0) = \frac{(-1)^{c+1}}{m_c z} \prod_{u_i} u_i \qquad F(z,1) = \frac{\prod_{u_i}(1-u_i)}{K(1,z)}$$

(product over the convergent roots u_i of the kernel K).

Proof:

$$f_{n+1}(u) = P(u) f_n(u) - \{u^{<0}\} P(u) f_n(u)$$

$$F(z,u)K(u,z) = \text{ LC of } \partial_u^j F(z,0)$$

$K(u,z) = 0 \Longrightarrow c$ convergent roots u_i

LC= polynomial (!) in u of degree c

factorization $\Longrightarrow$

$F(z,u) = \prod(u-u_i)/K(u,z)$ $\square$

Continued Fractions

natural (for steps -1,0,+1) by the traduction combinatorial structure $\leftrightarrow$ analysis :

$$\mathcal{E} = \mathrm{Seq}(\nearrow \mathcal{E} \searrow)$$

$$\Leftrightarrow E(z) = \frac{1}{1 - z^2 E(z)} = \cfrac{1}{1 - \cfrac{z^2}{1 - \ddots}}.$$

Links with orthogonals polynomials.

Arrangements $[2, (k) \rightarrow (k)(k+1)^{k-1}]$

$\exp(z)/(1-uz)$

Laguerre $[2, (k) \rightarrow (k+1)^{k-1}(k+2)]$

$\exp(u^2 z/(1-uz))/(1-uz)$

Hermite $[2, (k) \rightarrow (k-1)^{k-1}(k+1)]$

$\exp(uz + z^2/2)$

(involutions, marking fixed points)

Stirling $[2, (k) \rightarrow (k)^{k-1}(k+1)]$

$\exp(u\exp(z) - 1)$

Bessel $[0, (k) \rightarrow (k-1)(k)^k(k+1)]$

(permutations without croisements)

Euler $[1, (k) \rightarrow (k-1)^k(k+1)^k]$

$F(z,u) = \frac{\sin z + u\cos z}{\cos z - u \sin z}$

Generalized Dyck languages

alphabet: $(a_1, \ldots, a_m)$
each letter has a weight $(w_1, \ldots, w_m) \in \mathbb{Z}$
total weight of each word $= 0$

Theorem 5. *Generalized Dyck Languages are algebraic.*

Proof: Labelle and Yeh (90) & Duchon (98) make explicit the (cumbersome!) grammar $\square$

Our proof: easy! by the kernel method
Generalized Dyck words=excursions for the walk with the finite set of steps $(w_1, \ldots, w_m)$ whose GF is algebraic $\square$

Remark the two letters case $\Leftrightarrow$ walks under a rational slope line $(y = \frac{a}{b}x)$.

2-dimensional walks

Left to right jumps of knight coded by

$$P(u,v) := uv^2 + uv^{-2} + u^2v + u^2v^{-1}$$

We are rich $\Longrightarrow$ We have an infinite chessboard

$$F(z,u,v) = 1 + z\big(P(u,v)F(z,u,v)$$

$$-uv^{-1}\partial_v F(z,u,0) - (u^2v^{-1} + uv^{-2})F(z,u,0)\big)$$

"# paths from $(0,0)$ to $(k,0)$":

$$a_k = [u^k v^0]F(1,u,v).$$

$$F(1,u,0) = \frac{1+2u+\Delta}{4u} - \frac{\sqrt{2+2\Delta-8u^2+4u\Delta}}{4u}$$

$\Delta := \sqrt{1-4u+4u^2-4u^4}$ $\qquad a_k \sim (1+\sqrt{3})^k \approx 2.73^k \quad \square$

another approach: Labelle and Yeh (89)

Petkovšek knight problem

two jumps (-1,+2) or (+2,-1)

begins in (x, y) ($x = -1$ or -2, $y = -1$ or -2).

$a_{m,n} =$ # paths to (m, n)

$A(x, y) = \sum a_{m,n} x^m y^n \qquad F(u, z) = \sum f_n(u) z^n$

Variation on the Pascal triangle

$$\begin{array}{c}
1\\
1\quad 1\\
1\quad 1\quad \mathbf{1}\\
1\quad 1\quad \mathbf{1}\quad 1\\
1\quad 1\quad \mathbf{2}\quad 1\quad 1\\
1\quad 1\quad \mathbf{2}\quad 2\quad 1\quad 1\\
1\quad 1\quad \mathbf{3}\quad 2\quad 3\quad 1\quad 1\\
1\quad 1\quad \mathbf{3}\quad 4\quad 4\quad 3\quad 1\quad 1\\
1\quad 1\quad \mathbf{5}\quad 5\quad 6\quad 5\quad 5\quad 1\quad 1\\
1\quad 1\quad \mathbf{6}\quad 7\quad 10\quad 10\quad 7\quad 6\quad 1\quad 1\\
1\quad 1\quad \mathbf{8}\quad 11\quad 16\quad \underline{14}\quad 16\quad 11\quad \underline{8}\quad 1\quad 1\\
1\quad 1\quad \mathbf{12}\quad 17\quad 22\quad 27\quad 27\quad \underline{22}\quad 17\quad 12\quad 1\quad 1\\
1\quad 1\quad \mathbf{18}\quad 23\quad 39\quad 44\quad 44\quad 44\quad 39\quad 23\quad 18\quad 1\quad 1\\
1\quad 1\quad \mathbf{24}\quad 40\quad 62\quad 67\quad 83\quad 83\quad 67\quad 23\quad 18\quad 24\quad 1\quad 1\\
1\quad 1\ \mathbf{a_{12,2}}\qquad\qquad \ldots \qquad\qquad a_{2,12}\ 1\quad 1
\end{array}$$

Theorem 6 (Petkovšek). *F is irrational.*

Conjecture: F non D-finite ?

$$f_{n+1}(u) = (u^2 + \frac{1}{u})f_n(u) - \frac{f_n(0)}{u} + u - u[u^2]f_n - u^{n-1}[u^{n-3}] + u^{n-1} - u^{n+1}$$

$$F(z,u) = f_0 + f_1 z + f_2 z^2 +$$

$$z\left((u^2 + \frac{1}{u})(F - f_0 - f_1 z) - \frac{1}{u}(F(z,0) - 2) - uG(z) - \frac{1}{u}G(uz) + (\frac{1}{u} - u)(\frac{1}{1-uz} - 1 - uz)\right)$$

$$F(z,u)(1 - z(u^2 + \frac{1}{u})) = f_0 + f_1 z + f_2 z^2 + z\left((u^2 - \frac{1}{u})(f_0 + f_1) - \frac{1}{u}(\frac{1}{1-z} - 2) - u(G(z) - \frac{1}{u}G(uz) + (\frac{1}{u} - u)(\frac{1}{1-uz} - 1 - uz)\right)$$

$$F(z,u)K(u,z) = Rat(u,z) - uzG(z) - \frac{z}{u}G(uz)$$

$$1 - z(u^2 + \frac{1}{u}) = 0 \Longrightarrow G(zu_1) = Rat(u_1, z) - u_1^2 G(z)$$

$$G \circ \sigma = A + u_1^2 f$$

$$G\sigma^2 = A\sigma + (u_1\sigma)^2 f\sigma = A\sigma + (u_1\sigma)^2(A + u_1^2 G)$$

$$G\sigma^{n+1} = B\sigma_n \dots B\sigma B G + \sum_{i=0}^{n} A\sigma^i \prod_{i+1}^{n} B\sigma^j$$

Asymptotics of parameters for walks with finite jumps

paths	area	height	# of 0's
drift<0	$O(n)$	$O(\ln n)$	$O(n)$
drift$=0$	$O(n^{3/2})$	$O(\sqrt{n})$	$O(\sqrt{n})$
drift>0	$O(n^2)$	$O(n)$	$O(1)$

Average number of 0's

$$F(z,u,t) = \sum_{n,k,j} f_{nkj} z^n u^k t^j$$

$u \leftrightarrow$ height,$z \leftrightarrow$ length, $t \leftrightarrow$ number of 0's.

Example with Dyck words ($P(u) = u + \frac{1}{u}$):

$$F(z,u,t) = \frac{u^2(1-t)(u_1 - z) + zu_1}{((1-t)u_1 + zt)(zu^2 - u + z)}$$

Average # of 0's is

$$\mu_{2k} = \frac{[z^{2k}]\partial_t F(z,0,1)}{[z^{2k}]F(z,0,1)} = \frac{[z^{2k}]\frac{u_1(z)(u_1(z)-z)}{z^2}}{[z^{2k}]\frac{u_1(z)}{z}}$$

$$= \frac{\frac{\binom{2(k+1)}{k+1}}{k+2} - \frac{\binom{2k}{k}}{k+1}}{\frac{\binom{2k}{k}}{k+1}} = 3\frac{k}{k+2} = 3\sum_{p\geq 0} \frac{(-1)^p 2^p}{k^p}$$

$$= 3 - \frac{6}{k} + \frac{12}{k^2} + O(\frac{1}{k^3}).$$

$\square$

Admissibles functions

ϕ nonnegative coefficients, $\phi(0) \neq 0$,
ϕ nonconstant, R radius of CV of ϕ: either
$R = \infty$ either ϕ' has a pole in $z = R$

Theorem The radius of cv of u (where
$u^c = z\phi(u)$) is

$$\rho = \frac{\tau^c}{\phi(\tau)} = \frac{c\tau^{c-1}}{\phi'(\tau)}$$

Proof: singularity if cancelation of the leading term in $K(u, z) := u^c - z\phi$ or if

$$\begin{cases} K(u, \tau) = 0 \\ \frac{\partial K}{\partial_u}(u, \tau) = 0 \end{cases}$$

For $c \in [1, \deg(\phi)[$, $c = \frac{u\phi'(u)}{\phi(u)}$ has an unique solution τ. Indeed set $h := \phi(u)/u^c$,

$$h'' = \sum_{n=0}^{e} a_n(n-c)(n-c-1)x^{n-c-2} > 0$$

$h'(0+) < 0$ and $h(R) > 0$, Rolle theorem $\Longrightarrow$
$h' = 0$ has an unique solution on $]0, R[$. $\square$

Asymptotics of excursions

Lemma For $x > 0$

$$\phi(x)\left(x\phi''(x) + \phi'(x)\right) > x{\phi'}^2(x)$$

How many exursions of length n, for large n ?

Theorem 7. *Asymptotics* $\sim \frac{A}{\rho^n\sqrt{\pi n^3}}$. *$A$ and ρ (radius of convergence) are algebraic.*

Proof (by singularity analysis).
$F = \prod U_i$. But singulariti(es) of U_i dictate the growing rate of coeff of F.
U_i cumbersome but simple asymptotics: behaviour in $\sqrt{}$ around the singularity.

$$H(u) := \frac{\tau^c}{\phi(\tau)} - \frac{u^c}{\phi(u)}$$

$$\rho - z = H(u) = H(\tau) + \frac{1}{1!}H'(\tau)(u-\tau) + \frac{1}{2!}H''(\tau)(u-\tau)^2 + O((u-\tau)^3)$$

i.e. $\quad U - \tau = d_1\sqrt{\rho - z} + d_2(\rho - z) + \dots$

but $[z^n]\sqrt{1-z} = \frac{-1}{2\sqrt{\pi n^3}}(1 + O(\frac{1}{n}))$ $\square$

Technical lemma

ϕ nonnegative coefficients (nonconstant, $\phi(0) \neq 0$)
$\Longrightarrow \phi(x)(x\phi''(x) + \phi'(x)) > x\phi'^2(x)$

Proof: Note $\phi(x) = \sum_{n\geq 0} a_n x^n$.

$$\phi(x)(x^2\phi''(x) + x\phi'(x)) =$$

$$\Big(\sum_{n\geq 0} a_n x^n\Big)\Big(x^2 \sum_{n\geq 2} a_n n(n-1)x^{n-2} + x \sum_{n\geq 1} a_n n x^{n-1}\Big)$$

$$\sum_{n\geq 2} a_n n^2 x^n \geq \sum_{n\geq 1} 2n a_n x^n$$

$$\Big(\sum_{n\geq 2} a_n x^n\Big)\Big(\sum_{n\geq 2} a_n n^2 x^n\Big) \geq \Big(\sum_{n\geq 2} a_n n x^n\Big)^2$$

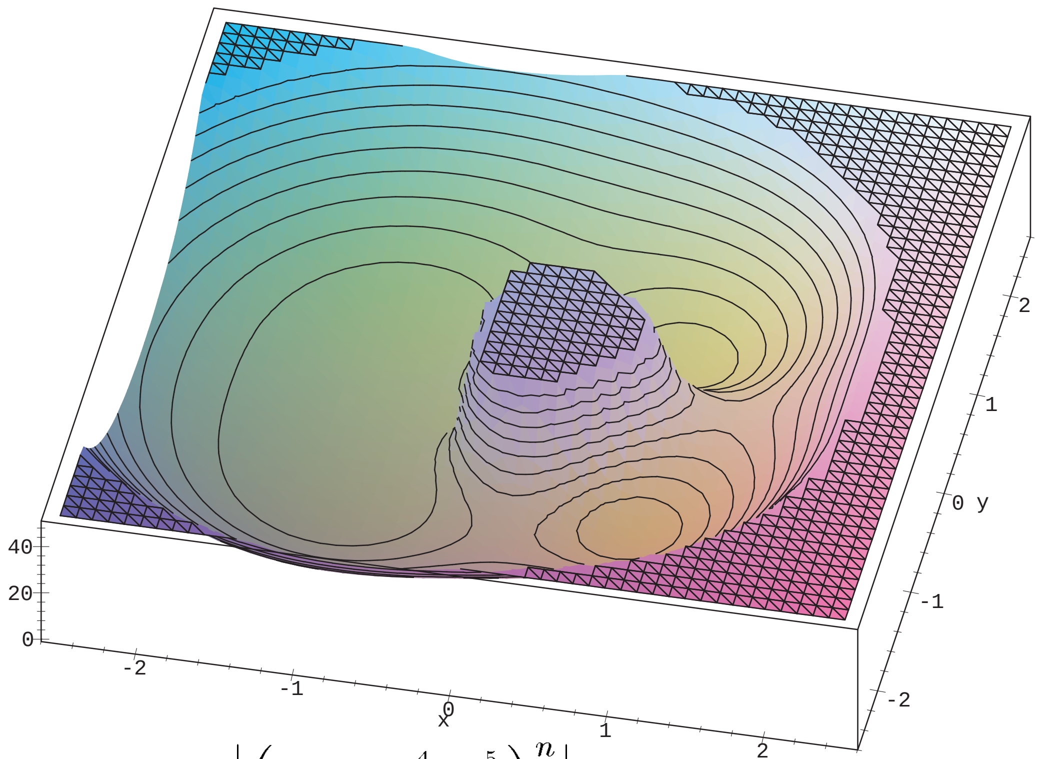
$$\sum_{k=2}^{n} a_{n-k}(n-k)^2 a_k \geq \sum_{k=2}^{n} a_{n-k}(n-k)a_k k.$$

boustrophedon:
$((n-k)^2 + k^2)a_{n-k}a_k \geq 2k(n-k)a_k a_{n-k}$,
ok because $(n-2k)^2 a_k a_{n-k} \geq 0$. $\square$

Another proof. Saddle point method.

Basic fact: coeff=integral, Cauchy integral formula: $f_n = \frac{1}{2i\pi}\int_\gamma \frac{F(z)}{z^{n+1}} dz$

Idea: play with the contour γ and obtain asymptotics for large n. $\square$



graph of $\left|\left(\frac{1+2u+u^4+u^5}{u^3}\right)^n\right|$, *i.e.* walk (-3,-2,-2,+1,+2).

For large n, all is concentrated around the saddle point.

Saddle Point Method

Prospection

“funny” questions:

• # of pairs (e_1, e_2) of excursions with e_1 always above e_2 (Fortin)

• difference of altitude of local extrema (Nicodème)

“asymptotics”

• limit laws (#0, height, area)

• model dynamic/static

• q-analog for area... (Airy?)

• determinant/symetrical functions for height

• link finite difference operators equation/ differential equation/ algebraic equation