The Local Limit Theorem for Random Walks on Free Groups

Steve Lalley
Purdue University

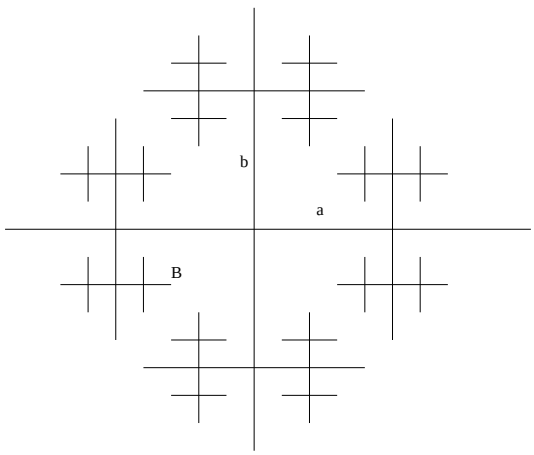
May 31, 1999

[summary by Cyril Banderier]

1. Introduction

Local limit theorems and saddlepoint approximations are given for random walks on a free group whose step distributions have finite support. These are derived by exploiting a set of algebraic relations among certain generating functions that arise naturally in connection with the transition probabilities of the random walks. Basic tools involved in the analysis are the elementary theory of algebraic functions, the Perron-Frobenius theory of nonnegative matrices, and standard techniques of singularity analysis.

2. Walks on Groups



A representation (Cayley graph) of the infinite free group $\mathbb{Z} \star \mathbb{Z}$.

Let $\mathbb{G}$ be a the free group with generators $a_1, \ldots, a_L$. For example, the free group $\mathbb{Z} \star \mathbb{Z}$ as two generators a and b. The word $aba\bar{a}\bar{b}\bar{a}\bar{a}\bar{b}aa$ corresponds to the B point, the reduced associated word is $\bar{a}\bar{b}aa$.

A finite-range random walk $\{Z_n\}_{n\geq 0}$ is a Markov chain on $\mathbb{G}$ with $Z_0 := e$ (the identity of the group, the "origin", the starting point) and transition probabilities

$$\text{Prob}\{Z_{n+1} = gx | Z_n = g\} = p_x \qquad \forall x, g \in \mathbb{G}, n \geq 0,$$

where p_x for $x \in \mathbb{G}$ is a probability distribution with finite support (in other words, p_x is the probability of the "jump" x).

Note $p^{*n}(x)$ the probability to be in x after n steps. It is assumed that the random walk is irreducible and aperiodic, that is, that

$$\text{Irreducibility:} \qquad \forall x \in \mathbb{G} \qquad \sum_{n \geq 1} p^{*n}(e) > 0;$$

$$\text{Aperiodicity:} \qquad \text{GCD}\{n \geq : p^{*n}(e) > 0\} = 1.$$

Another important condition is the following:

$$\text{Positivity:} \qquad p_e > 0 \text{ and } p_g > 0 \text{ for all the generators } g \text{ of } \mathbb{G} \text{ (and their inverses).}$$

Similarly to the random walk in the Euclidian case $\mathbb{Z}^d$, a local limit theorem is given by the asymptotics

$$p^{*n}(x) \sim \frac{B_x R^{-n}}{\sqrt{2\pi R} n^{3/2}}.$$

Similar results were already known when all the steps are of size one (*nearest neighbour* random walk [3]) or when all the words at a same distance from the origin are equiprobable (the so-called *isotropic* random walk [2, 8, 9]).

3. Analysis of Singularity

For $x \in \mathbb{G}$ and $z \in \mathbb{C}$ ($|z| < 1$), define

$$\text{the random variable coding where one is after } n \text{ steps:} \qquad Z_n,$$

$$\text{the PGF to reach } x \text{ in } n \text{ steps:} \qquad G_x(z) := \sum_n p^{*n}(x) z^n,$$

$$\text{the PGF of the excursions (Green's function):} \qquad G(z) := G_e(z),$$

$$\text{the first time } x \text{ is reached:} \qquad T_x := \inf\{n \geq 0 : Z_n = x\},$$

$$\text{the PGF to reach } x \text{ for the first time in } n \text{ steps:} \qquad F_x(z) := \sum_n \text{Prob}\{T_x = n\} z^n.$$

Remark: aperiodicity and irreducibility imply that for all sufficiently large $n \geq 1$, $p^{*n}(e) > 0$ and $p^{*n}(a) > 0$, for any (inverse of a) generator a.

$$G_x(z) = F_x(z) G(x)$$

$$G(z) = 1 + z \left(p_e + \sum_{x \neq e} p_x F_{x^{-1}}(z) \right) G(z) = \left(1 - z p_e - z \sum_{x \neq e} p_x F_{x^{-1}}(z) \right)^{-1}.$$

All of the functions F_x and G_x have the same radius of convergence R, $1 < R < \infty$.

Let $\mathbb{B}$ be the set of points at distance $\leq K$, where K is such that there is no smaller ball in which the support of the step distribution $\{p_x\}$ is contained. Define now

$$\text{the first time that } x \text{ is exceeded:} \qquad \tau_x$$

$$\text{the PGF to go from } a \text{ to } xb \text{ while } x \text{ as never been exceeded before:}$$

$$H_x^{ab} = \sum_n \text{Prob}\{Z_n = xb | Z_0 = a \text{ and } Z_{i<n} \notin x\mathbb{B}\} z^n$$

$$\text{the PGF to go from } x \text{ to the origin:} \qquad F_{x^{-1}}(z)$$

The formula

$$\forall x \neq e \qquad F_x(z) = \sum_{b \in \mathbb{B} - \{e\}} H_x^{eb}(z) F_{b^{-1}}(z)$$

leads to an expression of $F_x = uH_x(z)v = uH_{x_1}(z)\dots H_{x_m}(z)v$ where u is the e projection vector and v a vector whose inputs are the $F_{b^{-1}}(z)$ for $b \in \mathbb{B}$ and where the product of matrices is over $x_1 \dots x_m$ the reduced word associated to x.

It is then proven that all the non-constant $H_{x_i}^{ab}$ satisfy polynomial relations ($H_{x_i} = Q_i(H_{x_1}, H_{x_2}, \dots)$, see [6] for exact relations) and that they have the same radius of convergence.

By elimination (Groebner basis or resultants), the functions F_x and G_x are algebraic. Their Puiseux development leads to an algebraic singularity of coefficient $1/2$.

Here is a sketch of the proof that the exponent is indeed $\alpha = 1/2$. Define J_z the Jacobian matrix $(\partial Q_i \; \partial H_{x_i})$, the polynomial relations mentioned above are with nonnegative coefficients polynoms, thus there exist n such that J_z^n is an aperiodic and irreducible matrix with strictly positive coefficients. By the Perron-Frobenius theorem, J_z has a positive eigenvalue λ_z of multiplicity 1. The function λ_z is increasing and real-analytic and $\lambda_R = \frac{1}{R}$. Considering a left eigenvector of J_R and using the shape of the Q_i yields to the relations $(R-z)(C + \cdots) = C'(R-z)^{2\alpha} + \cdots$, thus $2\alpha = 1$.

As $z = R$ is the dominant singularity of F_x and G_x, one has the two following theorems:

Theorem 1 (Local limit theorem, access)**.** *Assuming irreducibility and aperiodicity, one has, for a positive constant B_x:*

$$p^{*n}(x) \sim \frac{B_x}{\sqrt{2\pi R} R^n n^{3/2}}.$$

Theorem 2 (Local limit theorem, first access)**.** *Assuming positivity, one has, for a positive constant A_x:*

$$\text{Prob}\{ \textit{ to reach } x \textit{ for the first time in } n \textit{ steps } \} \sim \frac{A_x}{\sqrt{2\pi R} R^n n^{3/2}}.$$

4. **Saddlepoint Approximations**

The probability to reach in n steps a point x at a distance m of the origin is

$$p^{*n}(x) \sim \frac{\exp(n\beta(m/n))}{\sqrt{\ddot{\psi}(m/n)}} C(m/n)$$

for appropriate functions β, C and $\ddot{\psi}$ (see the correct definitions / notations in [6]).

This uniform Asymptotics in x and $n$corresponds to the classical saddlepoint approximation (sharp large deviations theorems) for sums of iid random vectors in $\mathbb{R}^d$.

The saddlepoint approximations are of interest for another reason. For large n, nearly all the mass in the probability distribution $p^{*n}(x)$ is concentrated in the region $|x| \geq \epsilon n$, where the local limit approximations are not accurate. This contrasts with the situation for finite range random walk in Euclidean space. in fact, Guivarch [4] has shown that for random walks in $\mathbb{G}$, the distance from the origin grows linearly in n. Sawyer and Steger [10] have further shown that $(|Z_n| - n\beta)/\sqrt{n}$ converges in law to a normal distribution.

Finally, S. Lalley, using a peculiar matrix products and results on Ruelle's Perron-Frobenius operators, derives a saddlepoint approximation, uniformly for m/n in a given compact

$$\text{Prob}\{|Z_n| = m\} \sim \frac{\exp(nB(m/n))}{\sqrt{2\pi m D(m/n)}} C(m/n).$$

References

[1] Cartwright (Donald I.) and Sawyer (Stanley). – The Martin boundary for general isotropic random walks in a tree. *Journal of Theoretical Probability*, vol. 4, n° 1, 1991, pp. 111–136.

[2] Figà-Talamanca (Alessandro) and Picardello (Massimo A.). – *Harmonic analysis on free groups.* – Marcel Dekker Inc., New York, 1983, viii+145p.

[3] Gerl (Peter) and Woess (Wolfgang). – Local limits and harmonic functions for nonisotropic random walks on free groups. *Probability Theory and Related Fields*, vol. 71, n° 3, 1986, pp. 341–355.

[4] Guivarc'h (Y.). – Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire. In *Conference on Random Walks (Kleebach, 1979) (French)*, pp. 47–98, 3. – Soc. Math. France, Paris, 1980.

[5] Lalley (Steven P.). – Saddle-point approximations and space-time Martin boundary for nearest-neighbor random walk on a homogeneous tree. *Journal of Theoretical Probability*, vol. 4, n° 4, 1991, pp. 701–723.

[6] Lalley (Steven P.). – Finite range random walk on free groups and homogeneous trees. *The Annals of Probability*, vol. 21, n° 4, 1993, pp. 2087–2130.

[7] Lalley (Steven P.) and Hueter (Irene). – Anisotropic branching random walks on homogeneous trees. *Preprint*, 1999.

[8] Picardello (Massimo A.). – Spherical functions and local limit theorems on free groups. *Annali di Matematica Pura ed Applicata. Serie Quarta*, vol. 133, 1983, pp. 177–191.

[9] Sawyer (Stanley). – Isotropic random walks in a tree. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, vol. 42, n° 4, 1978, pp. 279–292.

[10] Sawyer (Stanley) and Steger (Tim). – The rate of escape for anisotropic random walks in a tree. *Probability Theory and Related Fields*, vol. 76, n° 2, 1987, pp. 207–230.

[11] Seneta (E.). – *Nonnegative matrices and Markov chains.* – Springer-Verlag, New York, 1981, second edition, xiii+279p.