Loop-erased random walks

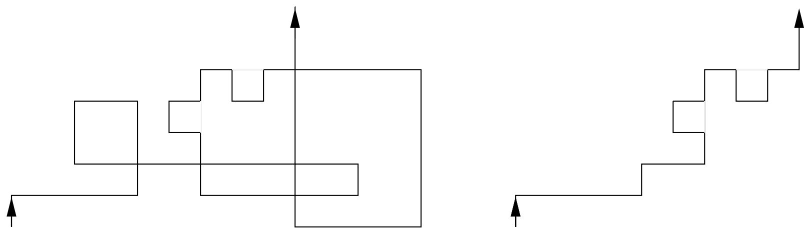
Richard Kenyon
Université de Paris-Sud

May 31, 1999

[summary by Cyril Banderier]

1. Introduction

The loop-erased random walk is the simple curve obtained by removing in the chronological order the loops of the original random walk. Two aspects of these walks in $\mathbb{Z}^2$ are studied: its average length (then solving a conjecture of Guttmann). Techniques are combinatorial and use a bijection (due to Temperley) between maximal trees and perfect coupling.



A random walk and its associated LERW

2. LERW=ST=DT

That's not a new equality between new complexity classes but simply a mark that loop-erased random walk (LERW), spanning tree (ST) and domino tiling (DT) are essentially the same object.

Indeed, in the article "Choosing a spanning tree for the integer lattice uniformly", Pemantle ([11]) shows that, for a uniformly chosen spanning tree on a region of $\mathbb{Z}^2$, the unique arc (branch) between two points has the same distribution as the LERW between these two points.

What's more, works of Temperley ([12]) gives a constructive one-to-one correspondence between spanning trees and domino tilings.

By the other talk of Richard Kenyon, we know that domino tiling (the so called two dimensional lattice dimer model, a model which has some ties with Ising model) is the only non ad hoc statistical physical model where conformal invariance is proved, so representations of Virasoro Algebra ([14]) could help to find some critical exponents.

In order to solve this "self-avoiding walk model" (*i.e.*, to set the critical exponent), R. Kenyon won't use representations theory, but a "discrete Laplacian", from which he gets the full asymptotics. Whereas a lot is know about properties of the continuous Laplacian ([4]), works on the discrete Laplacian are quite recent ([9]).

As conjectured by Bursill and Guttmann ([3]), the exponent for LERW is 5/4. Richard Kenyon proves this by applying the "equality" LEWR=ST on the following theorem

Theorem 1. *On the uniform spanning tree process on $\mathbb{N} \times \mathbb{Z}$, the expected number of vertices on the arc from $(0,0)$ to ∞ which lie within distance N of the origin is $N^{5/4+o(1)}$.*

The next theorem is sometimes attributed to Kirschoff ([10]) and proven in ([1])

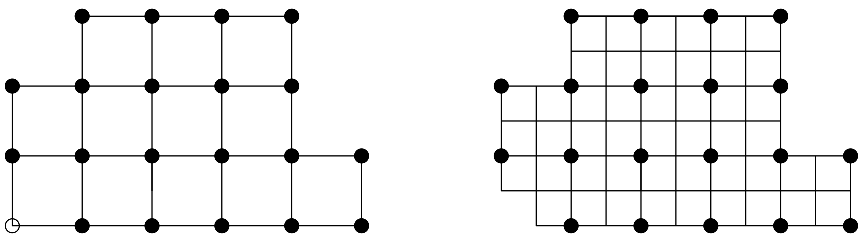
Theorem 2 (Matrix-tree Theorem). *For a graph G with a set of vertices $\{v_i\}$, let*

$$\Delta := \begin{pmatrix} \deg(v_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \deg(v_n) \end{pmatrix} - \mathrm{Adj}(G),$$

then number of spanning trees of G is the product of non zero eigenvalues of Δ divided by the size of G. For a $m \times n$ rectangle, the number of spanning trees is thus

$$\prod_{\substack{(j,k)\neq 0 \\ j=0,\ldots,m-1 \\ k=0,\ldots,n-1}} \left(4 - 2\cos(\frac{\pi k}{n}) - 2\cos(\frac{\pi j}{m})\right).$$

It is possible to extend this kind of result to a class of polygons which are decomposable in rectangles, the so-called Temperleyan polyominoes. (Triangulations are also a way, as there is a determinant-like expression for triangles.)



A graph and its associated temperleyan polyomino

The number of spanning trees of the graph is the number of domino tilings of its temperleyan polyomino. Taking the log in the previous formula gives a peculiar instance of the following theorem:

Theorem 3. *Let $U \subset \mathbb{R}^2$ be a rectilinear polygon with V vertices. For each $\epsilon > 0$, let P_ϵ be a Temperleyan polyomino $\in \epsilon\mathbb{Z}^2$ approximating U in the natural sense (the corners of p_ϵ are converging to the corners of U). Let A_ϵ be the area and $Perim_\epsilon$ be the perimeter of P_ϵ. Then the log of the number of domino tilings of P_ϵ is*

$$\frac{c_0 A_\epsilon}{\epsilon^2} + \frac{c_1 Perim_\epsilon}{\epsilon} + O(\log(\epsilon))$$

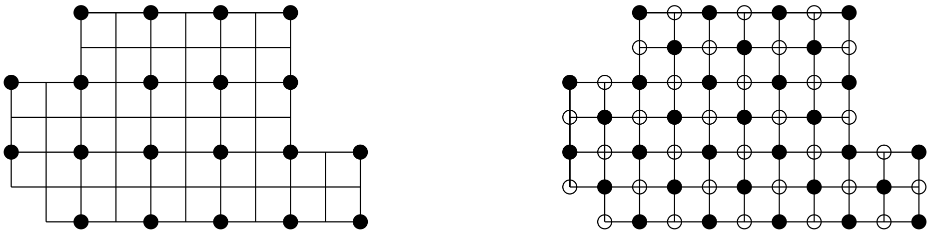
where $c_0 = \frac{G}{\pi}$, $G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \cdots$ is Catalan's constant, $c_1 = \frac{G}{2\pi} + \frac{\log(sqrt2-1)}{4}$.

According to the author, " Part of the motivation for the above theorem is to validate a certain heuristic, which attemps to explain how the presence of the boundary affects the long-range structure of a random tiling. In particular it attemps to explain how the boundary affects the densities of local configurations far from the boundary [dmb]. This heuristic is called the 'phason strain' principle. The heuristic is as follows: The boundary causes the *average height function* of a tiling to deviate slightly from its entropy-maximizing value of 0. At a point in the region where the average height function has nonzero slope, the "local" entropy there is smaller than the maximal possible entropy, by an amount proportional to the square of the gradient of the average height function. The system behaves in such a way as to maximise the total entropy subject to the given boundary values of the height function, and the resulting average height function is the function which minimises the (integral of) the square gradient. That is, the average height function is harmonic. "

Anyway an exact application of the phason strain principle gives in fact a slightly different asymptotics (as the one given in the theorem 3), so this principle is not totally valid here, but however it already gives a good approximation.

3. **The Height Function**

For a given tiling, the height function h is easily defined ([13]) by bicolouring the Temperleyan polyomino (there is no adjacent vertex of the same colour):



A temperleyan polyomino and the bicoloured associated graph

then, for each orientated edge AB (with AB on the border of the domino),

$$h(A) - h(B) := \begin{cases} +1 & \text{if the square on the left of AB is black} \\ -1 & \text{otherwise} \end{cases}$$

Note that, up to an arbitrary additive constant, the height of the boundary is independent of the tiling.

For a smaller and and smaller lattice (e.g. $\epsilon\mathbb{Z}^2$), you can approximate any domain U of $\mathbb{C}$. For a very fine lattice (in fact, take the limit when $\epsilon \to 0$), one can study the probability of appearance in the tiling of some patterns, and their repartition in the tiling and also how the shape of the boundary influences the tiling. It appears that there is some conformal mappings.

Let P_ϵ be an associated temperleyan polyomino of a rectilinear polygon U of $\mathbb{C}$. Now, let h_ϵ be the average height function of P_ϵ, that is the average height over all domino tilings of P_ϵ and then define

$$h(x) := \lim_{\epsilon \to 0} h_\epsilon(x_\epsilon)$$

For $x \in \partial U$, $h(x)$ is defined by continuity from values of h in the interior.

The remarkable fact is that this limiting average height function h can be expressed as

$$h(v) = \frac{4}{\pi}\Im \int_{b_0}^{v} \lim_{z \to v} \left(F_+(v,z) - \frac{2}{\pi(z-v)} \right) du = -\frac{2}{\pi}\Im \log \wp'(z),$$

where $\wp$ is the Weiertrass elliptic function and where $F_+(u,z)du$ is a meromorphic 1-form, allowing thus some links with conformal mappings/theory. We refer to "Conformal invariance of domino tiling" (1997) and to "The asymptotic determinant of the discrete Laplacian" (1999) for further informations.

Like other recent preprints of the author, they are available at his home page *http://topo.math.u-psud.fr/~kenyon/*

References

[1] Brooks (R. L.), Smith (C. A. B.), Stone (A. H.), and Tutte (W. T.). – The dissection of rectangles into squares. *Duke Mathematical Journal*, vol. 7, 1940, pp. 312–340.

[2] Burton (Robert) and Pemantle (Robin). – Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances. *The Annals of Probability*, vol. 21, n° 3, 1993, pp. 1329–1371.

[3] Guttmann (A.) and Bursill (R.). – Critical exponent for the loop-erased self-avoiding walk by monte-carlo methods. *J. Stat. Phys.*, vol. 59, 1990, pp. 1–9.

[4] Kac (Mark). – Can one hear the shape of a drum? *American Mathematical Monthly*, vol. 73, n° 4, part II, 1966, pp. 1–23.

[5] Kenyon (Richard). – Rigidity of planar tilings. *Inventiones Mathematicae*, vol. 107, n° 3, 1992, pp. 637–651.

[6] Kenyon (Richard). – Tiling a polygon with parallelograms. *Algorithmica*, vol. 9, n° 4, 1993, pp. 382–397.

[7] Kenyon (Richard). – Local statistics of lattice dimers. *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*, vol. 33, n° 5, 1997, pp. 591–618.

[8] Kenyon (Richard). – Tilings of convex polygons. *Annales de l'Institut Fourier*, vol. 47, n° 3, 1997, pp. 929–944.

[9] Kenyon (Richard). – Tilings and discrete Dirichlet problems. *Israel Journal of Mathematics*, vol. 105, 1998, pp. 61–84.

[10] Kirchhoff (G.). – Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird. *Ann. Phys. Chem.*, vol. 72, 1847, pp. 497–508.

[11] Pemantle (Robin). – Choosing a spanning tree for the integer lattice uniformly. *The Annals of Probability*, vol. 19, n° 4, 1991, pp. 1559–1574.

[12] Temperley (H. N. V.). – *Enumeration of graphs on a large periodic lattice.* – Cambridge Univ. Press, London, 1974, 155–159. London Math. Soc. Lecture Note Ser., No. 13p.

[13] Thurston (William P.). – Conway's tiling groups. *The American Mathematical Monthly*, vol. 97, n° 8, 1990, pp. 757–773.

[14] Virasoro (M.A.). – *Physical Review D. (3)*, 1970, pp. 2933–2936.