Some Sharp Concentration Results about Random Planar Triangulations

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[summary by Cyril Banderier]

Abstract

The theory of random maps has a relatively short history when compared to the theory of random graphs. In this talk, we mention some recent results concerning sharp concentration properties of parameters in random planar triangulations. Examples include the maximum vertex degree, the largest component, the number of copies of a given submap, and the number of flippable edges. This is joint work by Jason Zhicheng Gao and Nicholas Wormald (Melbourne, Australia).

1. Introduction

Draw a graph on a sphere and then mark (or *"root"*) a face, an edge of this face and a vertex of this edge. Then project the graph on the plane (e.g., with a stereographic projection): you get a *planar map*. Without loss of generality, you can rotate the sphere so that the marked face contains the north pole; then the projection will transform the marked face into an unbounded face, which is called the *external face*.

For background, we suggest reading the summary of Gao's talk (page ???-??? in these proceedings) for several definitions and examples on maps and triangulations.

For 10 years, Jason Zhicheng Gao (often collaborating with other specialists of maps, namely Bender, Canfield, Richmond, McKay, Wormald, ...) has studied several parameters (strong-connexity, pattern occurrences, vertex degree, symmetries, cycles, Eulerian properties) of maps, finding new functional equations, solving them, and also obtaining precise asymptotic estimations. We specialize here the discussion to two parameters: vertex degree and submap occurrence, which appear to be intimately related. Concentration results are obtained by the second moment method.

2. Submap density result

Many combinatorial structures satisfy *Borges' Theorem*[1], meaning that any pattern will appear with high probability in a large enough structure. Any word of length $\ln n$ appears with high probability in a random word of length n (for more details, see the study by Guibas and Odlyzko [6], and then by Nicodème et al. [7]). The occurrence of patterns in random graphs has also been studied [2]. However for maps, the situation is different as these objects live in quite a different probability space. Let's make a bet: choose a *map* of size 6, while I generate a random map of size 1000; would you bet that your map is a submap of mine? This talk makes explicit the conditions under which you can make good (or bad) bets.

[1]Philippe Flajolet coined this naming in reference to Borges's novel "The Library of Babel".

Let $\mathcal{T}$ be any fixed triangulation and $\eta_n(\mathcal{T})$ be the random variable for the *number of copies* of $\mathcal{T}$ in a random triangulation with n vertices. Richmond and Wormald (1988, see [8]) showed that

$$\Pr(\eta_n(\mathcal{T}) > cn) > 1 - \exp^{-\delta n}$$

for some positive constants c and δ depending on $\mathcal{T}$. Bender, Gao and Richmond (1992, see [1]) showed that the above result holds for many families of maps and recently Gao and Wormald (2000, see [5]) proved that $\eta_n(\mathcal{T})$ is sharply concentrated around cn for some constant c. More precisely:

Theorem 1. *Let $\mathcal{T}$ be a 3-connected triangulation with $j+3$ vertices such that there are r distinct ways to root $\mathcal{T}$. Let $c = 2r\left(\frac{27}{256}\right)^j$. Then, provided that $cn \to \infty$, $Pr(|\eta_n(\mathcal{T}) - cn| = o(cn)) \to 1$.*

A near-triangulation is composed of triangulations except that it can have more than 3 vertices on its external face. Define

$$\mu_k = \frac{8(k-2)}{4k^2-1}(-\frac{3}{4})^k\binom{-3/2}{k}.$$

Theorem 2. *Let $\mathcal{M}$ be a 3-connected near-triangulation with external face of degree k and with j internal vertices such that there are r distinct ways to root the external face. Then, for fixed j,k with $k \geq 4$, one has*

$$Pr\left(\left|\eta_n(\mathcal{M}) - r\mu_k\left(\frac{27}{256}\right)^{j-1} n\right| = o(n)\right) \to 1,$$

Proof. The method used here relies on Chebyshev-like inequalities: $\Pr(X > t\mu) \leq \frac{1}{t}$ and $\Pr(|X - \mu| \geq t\sigma) \leq \frac{1}{t^2}$ for a nonnegative random variable X with average μ and variance σ^2. A consequence is that if $\sigma = o(\mu)$, then one gets a concentration result.

First let us study the number $\zeta_n(k)$ of vertices of degree k in a random triangulation with $n+2$ vertices. The quantity T_n denotes the number of rooted triangulation with $n+2$ vertices; $T_{n,k}$ denotes the number of rooted triangulation with $n+2$ vertices and root vertex of degree k; $T_{n,k,l}$ denotes the number of rooted triangulation with $n+2$ vertices, root vertex of degree k and another distinguished vertex of degree l.

Step 1: Use combinatorial arguments to show

$$E\left[\zeta_n(k)\right] = \frac{6n}{k}\frac{T_{n,k}}{T_n} \qquad \text{and} \qquad E\left[\zeta_n(k)(\zeta_n(k)-1)\right] = \frac{6n}{k}\frac{T_{n,k,k}}{T_n}.$$

Step 2: Obtain functional equations for the generating functions for $T_{n,k,l}$, $T_{n,k}$ and T_n, and perform singularity analysis.

Step 3: Derive a suitable multivariate version of Flajolet and Odlyzko's transfer theorem (see lemmas 2 and 3 below), and obtain the following asymptotics, uniformly for $k = O(\ln n)$:

$$T_n = \sqrt{6}/(32\sqrt{\pi})n^{-5/2}(256/27)^n(1+O(1/n)),$$

$$T_{n,k} = \frac{k\sqrt{6}}{192\sqrt{\pi}}\mu_k n^{-5/2}(256/27)^n(1+O(k^{20}/n)),$$

$$T_{n,k,k} = \frac{k\sqrt{6}}{192\sqrt{\pi}}\mu_k^2 n^{-3/2}(256/27)^n(1+O(k^{20}/n)).$$

Step 4: Derive asymptotics for the first two moments of $\zeta_n(k)$;

$$E\left[\zeta_n(k)\right] = n\mu_k(1+O(k^{20}/n)) \qquad \text{and} \qquad \mathrm{Var}\left[\zeta_n(k)\right] = n\mu_k + (n\mu_k)^2 O(k^{20}/n),$$

uniformly for all $k = O(\ln n)$. It follows from Chebyshev's inequality that

$$\Pr\left(|\zeta_n(k) - \mu_k n| = o(\mu_k n)\right) \to 1$$

uniformly for all $k < (\ln n - (\ln\ln n)/2)\,/\ln(4/3) - \Omega(n)$. $\square$

Lemma 1. *Let $\mathcal{T}$ be a 3-connected near triangulation with $j+3$ vertices such that j is $o(n)$ and there are r distinct ways to root $\mathcal{T}$. Let $\eta_n(\mathcal{T})$ be the number of copies of $\mathcal{T}$ in a random rooted triangulation with $n+2$ vertices. Then*

$$E\left[\eta_n(\mathcal{T})\right] = r\left(\frac{27}{256}\right)^{j-1} E\left[\zeta_{n+1-j}(3)\right](1+o(1)),$$

$$E\left[\eta_n(\mathcal{T})(\eta_n(\mathcal{T})-1)\right] = r^2\left(\frac{27}{256}\right)^{2j-2} E\left[\zeta_{n+2-2j}(3)(\zeta_{n+2-2j}(3)-1)\right](1+o(1)).$$

In order to state precise results, one needs the following notation: ϵ is a small positive constant, ϕ is a constant satisfying $0<\phi<\pi/2$, and $\bar{y}=(y_1,y_2,\ldots,y_d)$. Define

$$\Delta_x(\epsilon,\phi) := \{x \text{ such that } |x|\leq 1+\epsilon, x\neq 1, |\mathrm{Arg}(x-1)|\geq\phi\}$$

$$\mathcal{R}(\epsilon,\phi) := \{(x,\bar{y}) \text{ such that } |y_j|<1, 1\leq j\leq d, x\in\Delta_x(\epsilon,\phi)\}.$$

Let $\beta_j>0$ for $1\leq j\leq d$, and α be any real number. The following two notations are also useful

Definition 1 ($\widetilde{O}$ notation)**.** We write $f(x,\bar{y})=\widetilde{O}\big((1-x)^{-\alpha}\prod_{j=1}^{d}(1-y_j)^{-\beta_j}\big)$ if there exist $\epsilon>0$ and $0<\phi<\pi/2$ such that, in $\mathcal{R}(\epsilon,\phi)$, $f(x,\bar{y})$ is analytic and $f(x,\bar{y})=O\big(|1-x|^{-\alpha}\prod_{j=1}^{d}(1-|y_j|)^{-\beta_j}\big)$ as $(1-x)(1-y_j)^{-p}\to 0$, for $1\leq j\leq d$, and some $p\geq 0$; for some $q\geq 0$ and some real number α', one has $f(x,\bar{y})=O\big(|1-x|^{-\alpha'}\prod_{j=1}^{d}(1-|y_j|)^{-q}\big)$.

Definition 2 ($\approx$ notation)**.** We write $f(x,\bar{y})\approx c(1-x)^{-\alpha}\prod_{j=1}^{d}(1-y_j)^{-\beta_j}$ if $f(x,\bar{y})$ can be expressed as $f(x,\bar{y})=c(\bar{y})(1-x)^{-\alpha}\prod_{j=1}^{d}(1-y_j)^{-\beta_j}+\sum_{j=0}^{d}C_j(x,\bar{y})+E(x,\bar{y})$ where $C_0(x,\bar{y})$ is a polynomial in x, and for $1\leq j\leq d$, $C_j(x,\bar{y})$ is a polynomial in y_j and where $E(x,\bar{y})=\widetilde{O}\big(|1-x|^{-\alpha'}\prod_{j=1}^{d}(1-y_j)^{-\beta'_j}\big)$ for some $\alpha'<\alpha$ and $\beta'_j\geq 0$, $1\leq j\leq n$ and finally where $c(\bar{y})=c+O(\sum_{j=1}^{d}|1-y_j|)$ and is analytic when $\bar{y}$ is componentwise in the $\Delta_{y_j}(\epsilon,\phi)$'s, with $c(\bar{1})=c\neq 0$.

Lemma 2. *Suppose that*

$$f(x,\bar{y})=\widetilde{O}\big((1-x)^{-\alpha}\prod_{j=1}^{d}(1-y_j)^{-\beta_j}\big);$$

then as $n\to\infty$ and $1\geq k_j=O(\ln n)$,

$$[x^n\bar{y}^k]f(x,\bar{y})=O\big(n^{\alpha-1}\prod_{j=1}^{d}k_j^{\beta_j}\big);$$

and for any $0<\epsilon'<1$ and for all n,k_j,

$$[x^n\bar{y}^k]f(x,\bar{y})=O\big(n^{\alpha-1}\prod_{j=1}^{d}(1-\epsilon')^{\beta_j}\big).$$

Lemma 3. *Let $d\geq 1$ and $f(x,\bar{y})\approx c.(1-x)^{-\alpha}\prod_{j=1}^{d}(1-y_j)^{-\beta_j}$, where α is neither a negative integer nor 0, and $c\neq 0$. Then as $n\to\infty$ and $k_j=O(\ln n)$,*

$$[x^n\bar{y}^k]f(x,\bar{y})=\frac{c}{\Gamma(\alpha)}\prod_{j=1}^{d}\left(k_j^{\beta_j-1}/\Gamma(\beta_j)\right)\left(1+O\big(\sum_{j=1}^{d}1/k_j\big)\right).$$

3. Maximum Vertex Degree

Let d_n be the maximum vertex degree of a random map in a family of maps of size n. Devroye, Flajolet, Hurtado, Noy and Steiger (1999, see [3]) showed that, for triangulations of an n-gon,

$$\Pr\left(|d_n - \ln(n)/\ln 2| \leq (1+\epsilon)\ln\ln n/\ln 2\right) \to 1.$$

Gao and Wormald [4] improved this last result and extended it to general families of maps. They showed that, for any function Ω going to infinity arbitrarily slowly, one has

- for triangulations of an n-gon: $\Pr\left(\left|d_n - \frac{\ln n + \ln\ln n}{\ln 2}\right| \leq \Omega(n)\right) \to 1$,
- for 3-connected triangulations of n vertices: $\Pr\left(\left|d_n - \frac{\ln n + (\ln\ln n)/2}{\ln(4/3)}\right| \leq \Omega(n)\right) \to 1$,
- for all maps of n edges: $\Pr\left(\left|d_n - \frac{\ln n + (\ln\ln n)/2}{\ln(6/5)}\right| \leq \Omega(n)\right) \to 1$.

4. Few open problems

At the end of the talk, few questions were raised, and the following conjectures appear plausible but might involve hard work.

Conjecture 1. The generating function of maps without a given submap is algebraic.

There is no doubt that a functional equation could be obtained for each pattern, (however, as the overlaps can be very intricate, the functional equation would be tremendous, and it is not sure that a generalization of the quadratic-method would allow to solve it, and prove algebraicity).

Conjecture 2. A Gaussian limit law should hold.

Once more, we expect the behavior to qualitatively be the same for words, trees and maps. Another interesting study which is as of now out of reach is the occurrence of a given pattern, not in a local sense but in a global one (such patterns are called "minors").

Other concentration results about random planar triangulations such as the largest component and the number of flippable edges were finally not presented in the talk but can be found in Jason Z. Gao's articles at his homepage `http://mathstat.math.carleton.ca/~zgao/`.

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