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Animals, Domino Tilings, Functional Equations

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Summary by Cyril Banderier

Abstract

This work lies within the framework of the great beat of animals in a square lattice: how to construct some new classes of animals, as large as possible, with enough structure to be exactly enumerable? It should be borne in mind that an animal is a finite connected set of vertices of a lattice (e.g., the square lattice), defined up to a translation, and that we still do not know the asymptotics of the number of animals with n vertices.

Our starting point is a correspondence, due to Viennot, between directed animals and pyramids of dominoes. We define a (much) larger class of animals, in one-to-one correspondence with some so-called connected domino tilings, and we proceed to their enumeration.

To this aim, we have to solve a functional equation, a variant of which gives the generating functions of directed animals. The two models are however quite distinct: directed animals have an algebraic generating functions, and a growing constant equal to 3, whereas our new class of animals has a non D-finite generating function and a growing constant of 3.58.

We can say that we did half the journey until the animal Graal: their growing constant is estimated to 4.06... (joint work with Andrew Rechnitzer).

One of the most celebrated open problems in combinatorics is the enumeration of animals (also called polyominoes). A polyomino of area n is a connected union of n cells on a lattice (symmetries are not taken into account: e.g., there are two polyominoes of area 2). Animals can be seen as duals of a polyominoes, with each cell replaced by a vertex at its centre.

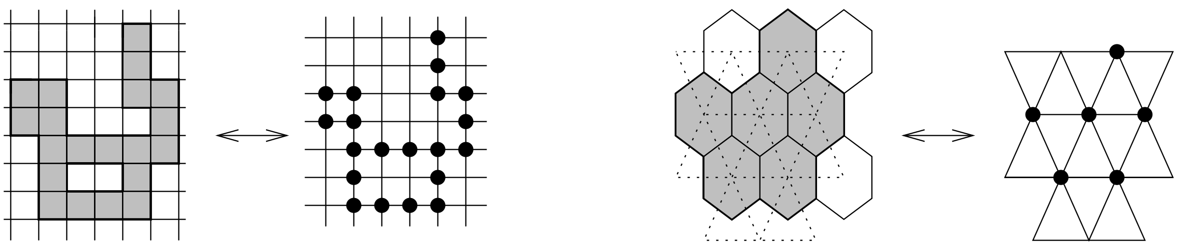


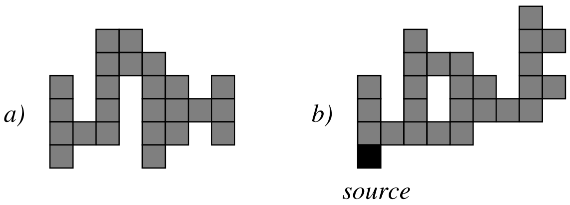
FIGURE 1. Polyominoes with square and hexagonal cells, and the corresponding animals on the square and triangular lattices.

Since the 1950's, combinatoricians and physicists (as animals are related to percolation models) tried without success to get a nice formula for the number of animals of size n or to make their asymptotics explicit.

Let a_n be the number of animals of size n on the square lattice. A concatenation argument due to Klarner implies that a_n has an exponential growing rate, i.e., $a_n^{1/n}$ converges to a constant μ (called

Model	μ	Nature of the GF	Who solved it (first)
Rectangles	1	q-series	obvious
Ferrers diagrams (partitions)	1	q-series	Euler 1748
Stacks	1	q-series	Auluck 1951, Wright 1968
Parallelogram	2.30...	q-series	Klarner & Rivest 1974
Directed convex	2.30...	q-series	Bousquet-Mélou & Viennot 1992
Convex	2.30...	q-series	Bousquet-Mélou & Fédou 1995
Bargraph (compositions)	2	rational	obvious
Directed column convex	2.62...	rational	Moser, Klarner 1965
Column convex	3.20...	rational	Temperley 1956
Directed	3	algebraic	Dhar 1982
Stacked directed	3.5	algebraic	Bousquet-Mélou & Rechnitzer 2002
Multi directed	3.58...	non D-finite	Bousquet-Mélou & Rechnitzer 2002
General	4.06?	???	You?

TABLE 1. Some of the solved subclasses of square lattice polyominoes and their growth constants.



A polyomino is *column-convex* if its intersection with any vertical line is connected; it is *directed* if any cell can be reached from a fixed cell called the *source*, by a North-East directed path that only visits cells of the animal.

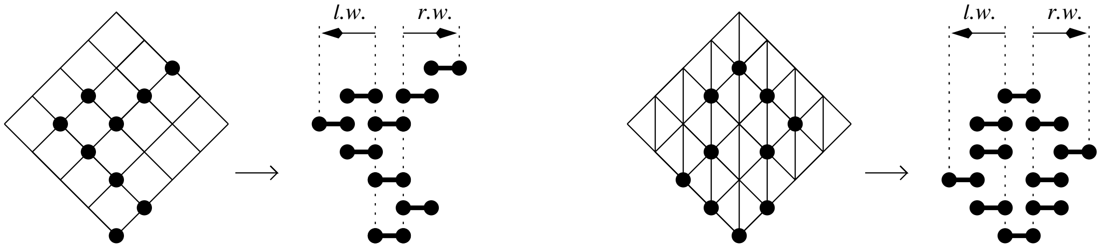


FIGURE 2. A column-convex polyomino (a) and a directed polyomino (b). The second line illustrates that replacing each vertex of an animal by a dimer transforms a directed animal into a "pyramid" (a heap of dimers).

Klarner's constant). Numerical studies suggest that $a_n \approx C\frac{4.06^n}{n}$. The first 46 terms $a_1, \ldots, a_{46}$ have been computed;[1] it begins like: 1, 2, 6, 19, 63, 216, 760, 2725, 9910, ... As a byproduct (via Klarner's concatenation argument), it implies that $3.9 < \mu < 4.65$.

As is usual, people tried to solve simpler problems which were more or less direct simplification of the general model. Progress were done by adding some convexity or directedness constraints—see Table 1.

[1] See Steve Finch's website on constants at `http://algo.inria.fr/bsolve/constant/constant.html` for up-to-date datas on the Klarner's constant.

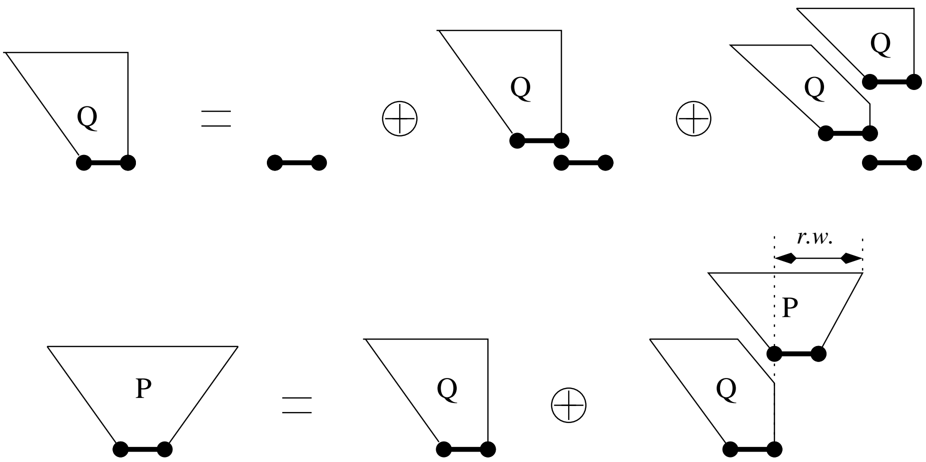


FIGURE 3. The first line shows that a half-pyramid Q factorizes in smaller half-pyramids. The second line shows that a pyramid P can be factorized in terms of half-pyramids Q and a smaller pyramid.

The Schützenberger methodology (also called "the symbolic method") classically gives the generating functions of combinatorial structures which factorize. For pyramids (heap of dimers), the factorization in Figure 3 gives a system of functional equation $P(x) = Q(x) + Q(x)P(x)$ and $Q(x) = x + xQ(x) + xQ(x)^2$, solving it gives the generating functions of half-pyramids

$$Q(x) = \frac{1 - x - \sqrt{(1+x)(1-3x)}}{2x}.$$

From this, the bivariate generating function of pyramids (x encoding the number of dimers and w encoding the right width) is $P(x,w) = Q(x)\big(1 - uQ(x)\big)^{-1}$. This gives $\mu = 3$ for directed animals (see Table 1), and also that their average width (which is given by twice the *right* width plus one) is asymptotically $6\sqrt{3\pi n}$.

We now define in the following figure two new classes of animals: stacked directed animals and multidirected animals. (See examples on Figure 4.)

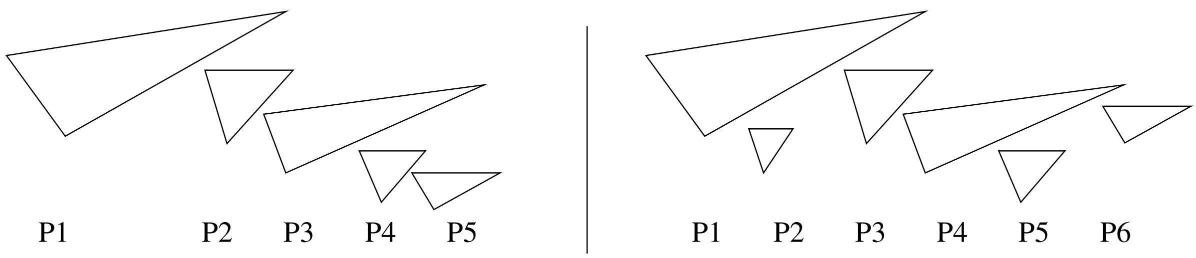


FIGURE 4. Each triangle represents a directed animal. In stacked directed animal (left) each directed animal component, P_i lies below P_{i-1}, whereas in a multi-directed animal (right) P_i lies below P_j for some $j < i$. So the right drawing is not a stacked directed animal as P_3 is *above* P_2.

We have already computed $P(x, v)$, the generating function for pyramids. The generating function $S(x, w, t)$ for stacked directed animals (x enumerates the number of dimers, w the right width and t the number of sources) is algebraic and given by $S(x, w, t) = t\frac{P(x,w)}{1-tP(x,1)^2}$. This comes easily from the functional equation

$$S(x, w, t) = tP(x, w) + tP(x, w)\partial_w S(x, 1, t)$$

which reflects the fact that a stacked animal is either a single pyramid or a pyramid with another stacked animals "attached" below it. There are r ways to attach it, if r is the length of the pyramid; this "attachement" (or "pointing") is translated by a differentiation with respect to w in the functional equation.

The generating function for multi-directed animals is

$$M(x) = \frac{Q}{(1-Q)\left(1 - \sum_{k\geq 1} \frac{Q^{k+1}}{1-Q^k(1+Q)}\right)}.$$

Consequently, $M(x)$ is not D-finite;[2] this comes from the fact that the zeroes of $1 - q^n(1 + q)$ accumulate on a part of the circle $|q| = 1$, whereas a D-finite function has only finitely many singularities.

The generating function is in fact obtained by $M(x) = C(x, x, 1)$, where $C(x, y, w)$ is the generating functions of connected heaps (x encodes the number of dimers, w the width and y the size of the rightmost column) and satisfies the following functional equation

$$C(x, y, w) = \frac{uy}{1-y} + \frac{u}{1-y}C\left(x, \frac{x}{1-y}, w\right) - wC(x, x, w).$$

Iterating this recursive definition leads to

$$1 + C(x, y, w) = \left(\sum_{n\geq 0} \frac{u^n}{F_n(x) - yF_{n-1}(x)}\right) - \left(\sum_{n\geq 1} \frac{u^n}{F_n(x)}\right)(1 + C(x, y, w))$$

which is equivalent to

$$C(x, y, w) = -1 + \left(\sum_{n\geq 0} \frac{u^n}{F_n(x) - yF_{n-1}(x)}\right)\left(\sum_{n\geq 0} \frac{u^n}{F_n(x)}\right)^{-1} \tag{1}$$

where $F_n(x)$ stands the nth Fibonacci polynomial, defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} - xF_{n-2}$.

It is interesting to note that from formulas similar to (1), one gets that some other generating functions $R(x, y, w)$ are non D-finite. The proof relies on the fact that these generating functions involve

$$\sum_{n\geq 1} \frac{q^n}{1-q^n} = \sum_{n\geq 1} d(n)q^n \qquad \text{or} \qquad \sum_{n\geq 1} \frac{q^n}{(1-q^n)^2} = \sum_{n\geq 1} \sigma(n)q^n$$

where $d(n)$ is the number of divisors of n and $\sigma(n)$ the sum of the divisors of n, two well-known functions of number theory. Evaluating these functions modulo 2 relates them to the generating functions of square numbers $\sum_{n\geq 1} z^{n^2}$, which is not rational (either as a lacunary series, either by p-automatic considerations). But a series with integer coefficients and with radius 1, is either rational or has the circle as natural boundary (Fatou–Pólya–Carlson theorems). So $R(x, y, w)$ does not have finitely many singularities and hence is not D-finite.

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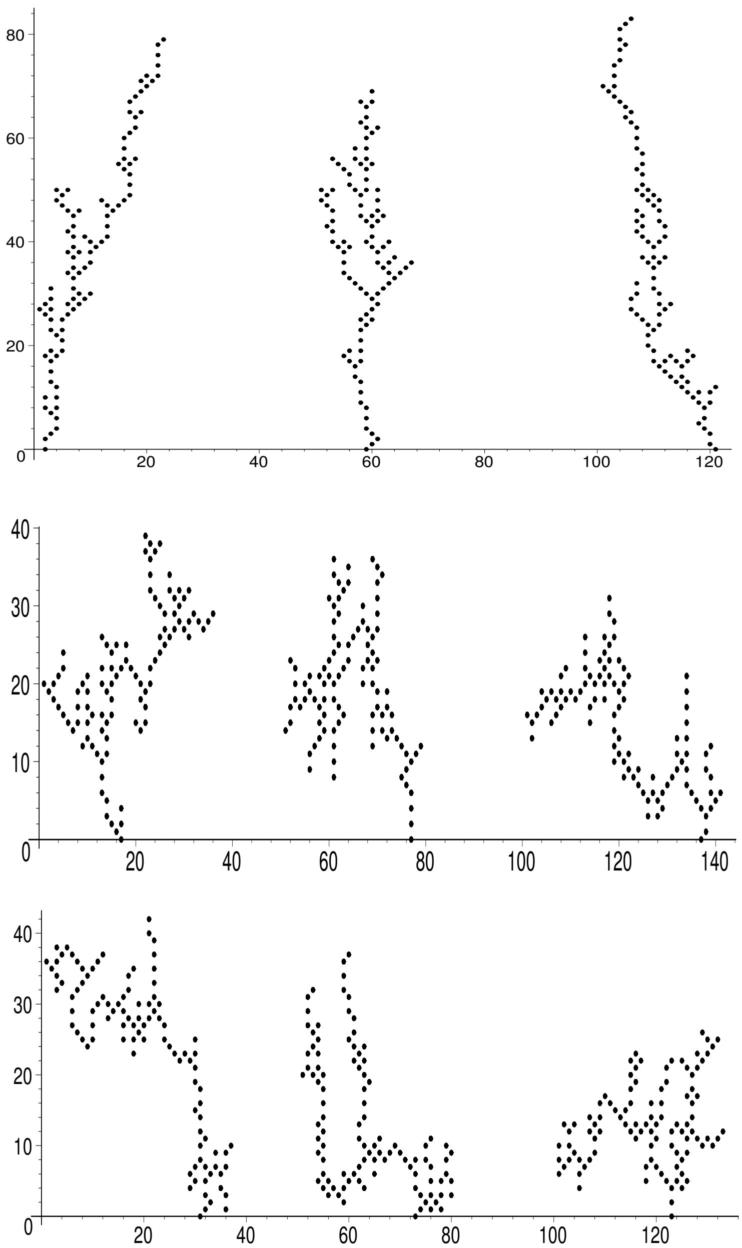
[2]Recall that a function $F(x)$ is called D-finite whenever there are some polynomials $p_i(x)$ and an integer d such that $p_d(x)\partial^d F(x) + \cdots + \partial F(x) + p_1(x)F(x) + p_0(x) = 0$. This is an important class of generating functions, very well suited to computer algebra methods.



3 directed animals.
(width= $O(\sqrt{n})$ in average)

3 stacked directed animals.
(width= $O(n)$ in average)

3 multi-directed animals.
(width = $O(n)$ in average)

FIGURE 5. Pictures of animals drawn uniformly at random amongst animals of size 100 (Mireille's zoo).

From a prospective viewpoint, it is perhaps possible to extend this approach to more sophisticated structures of animals (e.g., partially directed animals). The nature of the generating function of general animals/polyominoes remains an open problem.

This small note is a summary of M. Bousquet-Mélou & A. Rechnitzer article [1], available online at `http://dept-info.labri.u-bordeaux.fr/~bousquet/`.

Bibliography

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