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Lattice paths with an infinite set of jumps

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A model for random walks

The walk starts at time 0 from a point Σ_0 of $\mathbb{Z}$; at time n, one makes a jump $X_n \in \mathbb{Z}$ and the new position is given by the recurrence $\Sigma_n = \Sigma_{n-1} + X_n$ where, when $\Sigma_{n-1} = k$, the X_n's are constrained to belong to a set $\mathcal{P}_k$.

These walks are homogeneous in time: the set of jumps when one is at position k is independent from the time.

- the sets $\mathcal{P}_k$ are equal to a fixed set $\mathcal{P}$: the walks have been deeply studied and can be generated by a context-free grammar (e.g., $\mathcal{P} = \{-1, +1\}$). These walks are homogeneous in space: the set of available jumps doesn't depend on the position.

- the sets $\mathcal{P}_k$ are unbounded: contrary to the previous case, the walks are not space-homogeneous and it is not possible to generate them by context-free grammars.

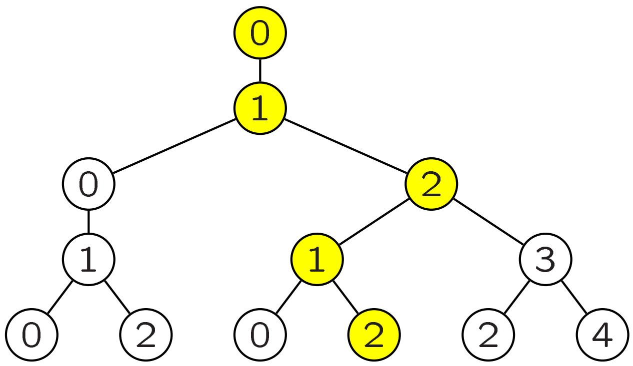
Lattice paths and generating trees

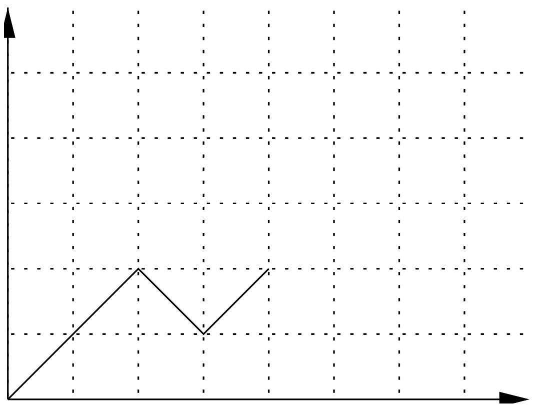
In combinatorics, it is classical to represent a particular walk of length n as a path in a two dimensional lattice.

It is also convenient to represent all the walks of length $\leq n$ as a tree of height n.

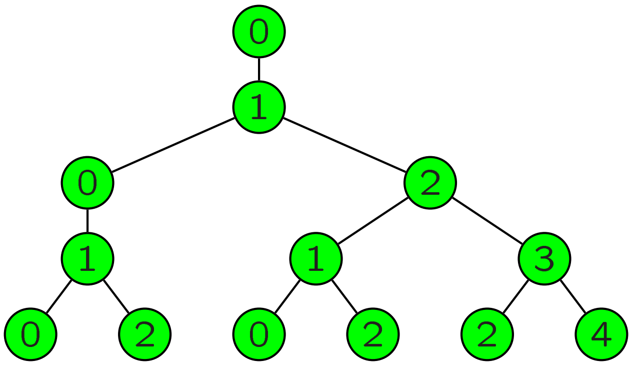
Example

The generating tree of the walk on $\mathbb{N}$ with jumps $\mathcal{P} = \{+1, -1\}$ starting in 0 and up to length $n = 4$.





Generating trees and rewriting rules



$$[(0), (k) \rightsquigarrow (k-1)(k+1)]$$

Matrix associated to a generating tree: is an infinite matrix $\{d_{n,k}\}_{n,k \in \mathbb{N}}$ where $d_{n,k}$ is the number of nodes at level n with label $k + r$, r being the label of the root:

n/k	0	1	2	3	4
0	1				
1	0	1			
2	1	0	1		
3	0	2	0	1	
4	2	0	3	0	1

= Henri Delannoy triangular chessboards (1886).
= Riordan arrays [7,10,11].

Walks on $\mathbb{Z}$ with an infinite set of negative jumps

Consider a sequence $(e_i(k))_{i\geq -a}$ (for a given integer $a>0$) of polynomials assuming nonnegative integers values:

$$[(r),(k) \rightsquigarrow (0)^{e_k(k)}(1)^{e_{k-1}(k)}\dots(k)^{e_0(k)}\dots(k+a)^{e_{-a}(k)}]$$

The exponent $e_i(k)$ is the multiplicity of the jumps $-i$ when one is at position k and r is the starting position of the walk (or equivalently, the root of the associated generating tree)

1) If the sequence of polynomials $(e_i(k))_{i\geq -a}$ is ultimately $e_i(k)=0$, then we have walks with a finite set of jumps.

2) If the sequence is ultimately $e_i(k)=1$, then we have "factorial rules" which are of great interests for the generation of combinatorial objects (see, e.g., [4]) and for which it was proven in [2] that the associated generating functions are algebraic.

Rationality and algebraicity of classes of rewriting rules

Let $f_{n,k}$ be the number of walks on $\mathbb{N}$ of length n going from the starting point to k and let $F(z,u) = \sum_{n,k\geq 0} f_{n,k} u^k z^n$.

Theorem 1: *For any constant $B \geq 0$, the rule*

$$[(r),(k) \rightsquigarrow (0)^{e_k(k)} \dots (B)^{e_{k-B}(k)} (k)^{e_0} \dots (k+a)^{e_{-a}}]$$

(where $e_k(k), \dots, e_{k-B}(k)$ are polynomial in k, $e_i(k) = 0$ for $0 < i < k-B$ and $e_i(k) = e_i$, some fixed constants, for $i \leq 0$) has a rational generating function $F(z,u)$.

Example: $B = 3$, $e_k(k) = k^2, e_{k-1} = 0, e_{k-2} = 3k-1, e_{k-3} = 1$, $e_0 = 1$, $e_{-1} = 2$, $e_{-2} = 0$, and $e_{-3} = 5$:

$$[(0),(k) \rightsquigarrow (0)^{k^2}(2)^{3k-1}(3)\,(k)(k+1)^2(k+3)^5]$$

$$F(z,u) = \frac{u^3(22z^2 - 112z^3 - z) + u^2(480z^3 - 60z^2) + 528z^3 - 250z^2 + 31z - 1}{(1 - zP(u))(872z^3 - 212z^2 + 30z - 1)}$$

Theorem 2: *For a constant $B \geq 0$, the rule*

$$[(r),(k) \rightsquigarrow (0)^{e_k(k)} \ldots (B)^{e_{k-B}(k)} (B{+}1) \ldots (k{-}b{-}1)(k{-}b)^{e_b} \ldots (k{+}a)^{e_{-a}}]$$

(where $e_k(k), \ldots, e_{k-B}(k)$ are polynomial in k, $e_i(k) = 1$ for $b < i < k - B$ and $e_i(k) = e_i$, some fixed constants, for $i \leq b$) has an algebraic generating function $F(z,u)$.

Example: $B = 5, b = 4, a = 23$, $e_k(k) = k^2$, $e_{k-2}(k) = 3k^5 - 2$, $e_{k-1}(k) = e_{k-3}(k) = e_{k-4}(k) = e_{k-5}(k) = 0$, $e_4 = 2$, $e_2 = 3$, $e_0 = 1$, $e_{-3} = 2$ and $e_{-23} = 1$:

$$[(0),(k) \rightsquigarrow (0)^{k^2}(2)^{3k^5-2}(6)(7)\ldots(k{-}5)(k{-}4)^2(k{-}2)^3(k)(k{+}3)^2(k{+}11)]$$

Theorem 3: *Consider the rule*

$$[(0),(k) \rightsquigarrow (0)^{e_k}(1)^{e_{k-1}}\dots(k-1)^{e_1}(k)^{e_0}\dots(k+a)^{e_{-a}}]\,.$$

If the generating function of the exponents $E(u)$ is algebraic then the bivariate generating function of the walk $F(z,u)$ is algebraic. For $a=1$, one has

$$F(z,u)=\frac{F_0(z)}{1-u\,e_{-1}z\,F_0(z)} \qquad F_0(z)=\frac{1}{e_{-1}z}E^{<-1>}(\frac{1}{z})$$

where $E^{<-1>}$ is the compositional inverse of $E(u)$ and where e_{-1} is the multiplicity of the $+1$ jump. More generally, for $a\geq 1$, the generating function $F(z,u)$ is expressed in terms of the a solutions $u_1(z),\dots,u_a(z)$ of $1-zE(u)=0$ which satisfy $u_i(z)\sim 0$ for $z\sim 0$:

$$F(z,u)=F_0(z)\prod_{i=1}^{a}\frac{1}{1-uu_i(z)}=$$

$$=\sum_{k\geq 0}F_0(z)\left(\sum_{i_1+\ldots+i_a=k}u_1^{i_1}\dots u_a^{i_a}\right)u^k\,.$$

One has

$$F_0(z)=\frac{(-1)^{a+1}}{ze_{-a}}\prod_{i=1}^{a}u_i(z) \qquad F(z,1)=\frac{-1}{ze_{-a}}\prod_{i=1}^{a}\frac{1}{1-\frac{1}{u_i(z)}}\,.$$

The proof is via the Kernel Method.

Some remarks on Theorem 3

The case $a = 1$ corresponds to a theorem in [10] which proves that if $a_j, z_j \in \mathbb{N}$, $a_0 \neq 0$, then the the matrix associated to the rule

$$[(0), \{(k) \rightsquigarrow (0)^{z_k}(1)^{a_k}(2)^{a_{k-1}} \dots (k)^{a_1}(k+1)^{a_0}\}]$$

is a proper Riordan array D defined by the triple

$$(1, A = (a_0, a_1, \dots), Z = (z_0, z_1, \dots))$$

and vice versa.

A *proper Riordan array* (see, e.g. [7,11]) is an infinite lower triangular array $D = \{d_{n,k}\}_{n,k \in N}$ for which two sequences $A = \{a_0, a_1, a_2, \dots\}$, $a_0 \neq 0$ and $Z = \{z_0, z_1, z_2, \dots\}$, exist such that:

$$d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}, \qquad d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}.$$

The Riordan arrays approach immediately gives the algebracity of $F(z, u)$ when $A(z)$ and $Z(z)$ are algebraic. In fact, for a Riordan array we have

$$F(z, u) = \frac{d(z)}{1 - uzh(z)}$$

where $h(z) = A(zh(z))$ and $d(z) = 1/(1 - zZ(zh(z)))$. When $A(z) = d(0) + zZ(z)$, that is $z_k = a_{k+1}$ in the rule, we have $d(z) = h(z)$ and in this case the function $F_0(z)$ in the statement of the theorem coincides with $h(z)$.

In general, the algebracity of the generating function corresponding to the rule can be approached by using the concept of a *horizontally stretched Riordan array*.

Ex. 1:The s-tennis ball problem

At the first turn, you are given balls labelled 1, $2, \ldots, s$. You throw one of them out of the window onto the lawn.

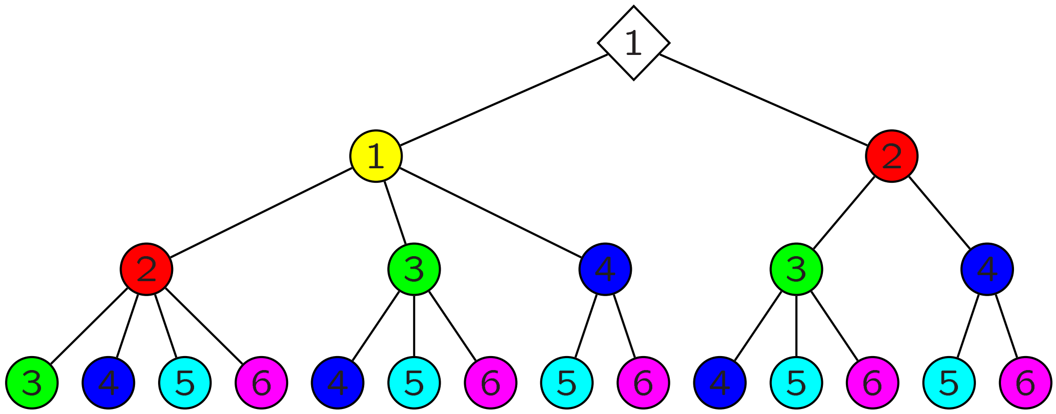
At the second turn, balls labelled $s+1, s+2, \ldots 2s$ are brought in. You throw out on the lawn any of the $2s-1$ remaining balls.

Then balls $2s+1, 2s+2, \ldots, 3s$ are brought in. You throw out one of the $3s-2$ available balls.

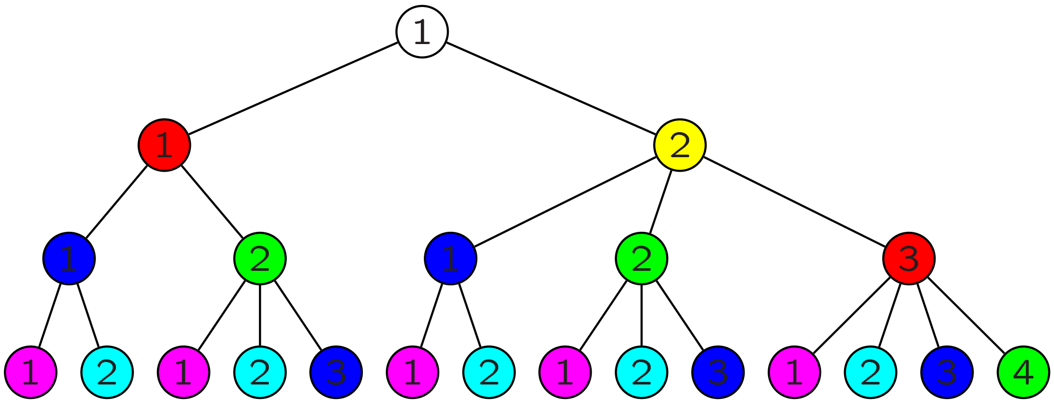
The game continues for n-turns. The ordered sequence of the balls on the lawn is a *tennis ball s-sequence*.

How many tennis ball s-sequences of length n? What is the sum of all the balls in all the possible s-sequences of length n? (solved in [8]).

An s-sequence of length n is just a path in a tree. For example, in the case $s = 2$:



$$[(1),(k) \rightsquigarrow (1)\cdots(k)(k+1)]$$



The two trees are isomorphic.

In general, if we fix the number s, we get generating tree corresponding to the rule:

$$[(1),(k) \rightsquigarrow (1)\cdots(k+s-2)(k+s-1)]$$

The number of tennis ball s-sequences of length n are counted by T_{n+1}, where

$$T_n = \frac{1}{1+(s-1)n}\binom{sn}{n}$$

The cumulative sum of all the balls thrown onto the lawn in n turn is

$$\Sigma_n = \frac{1}{2}(sn^2+(3s-1)n+2s)T_{n+1} -$$

$$\frac{1}{2}\sum_{k=0}^{n+1}\binom{sk}{k}\binom{s(n+1-k)}{n+1-k}.$$

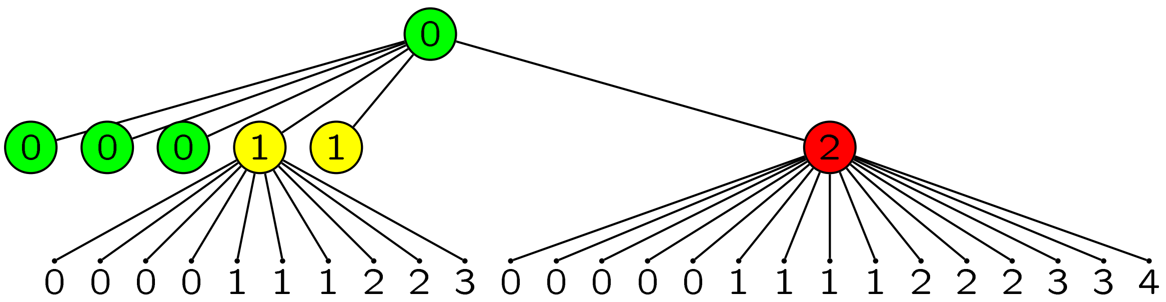
Ex. 2: a new rewriting rule for the $(4,2)$-tennis ball problem

Consider the following variant of the problem of balls: one receives s balls at each turn but now throws out t balls at a time.

General (s,t) case = open problem $(s > t \geq 2)$.

For the $(4,2)$ case, Merlini and Sprugnoli found the following rule:

$$[(0),(k) \rightsquigarrow (0)^{k+3}(1)^{k+2}(2)^{k+1}\dots(k+2)]$$

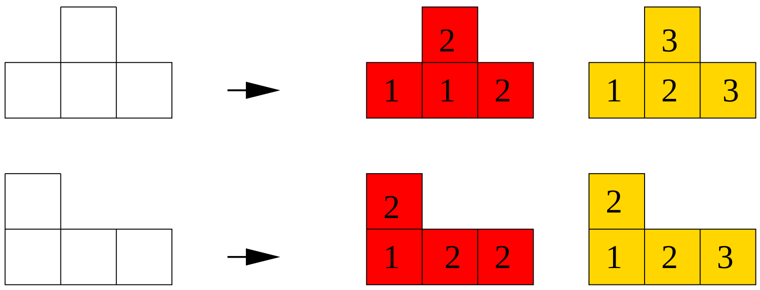


Ex. 3: a combinatorial model for job scheduling

In [9] the authors present a combinatorial model for studying the characteristics of job scheduling in a slow device, for example a printer in a local network.

The policy usually adopted by spooling systems is called *First Come First Served* (FCFS) and can be realized by queuing the processes according to their arrival time and by using a FIFO algorithm. A job (printing a file) consists in a finite number of *actions* (printing-out a single page). Each action takes constant time to be performed (a *time slot*).

If we fix n time slots, and suppose that at the end of the period the queue becomes empty, while it was never empty before, the successive states of the jobs queue can be described by a combinatorial structure called *labelled 1-histograms*.

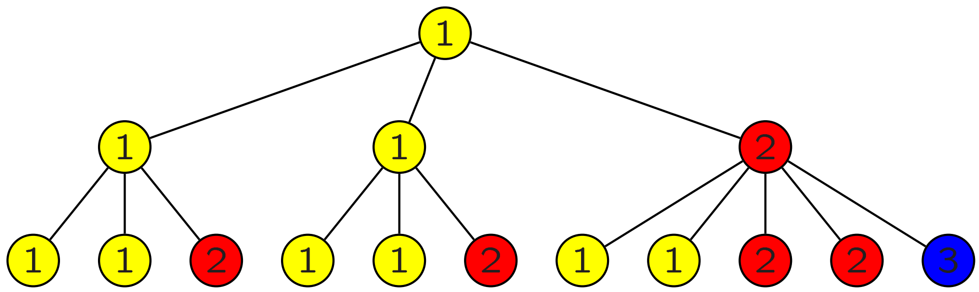


It can be proved that the number of schedules of length n with k jobs request at the first time slot corresponds to the number of nodes at level n having label $k+1$ in the generating tree with specification:

$$[(1), (k) \rightsquigarrow (1)^2 \dots (k)^2 (k+1)] \, .$$

This gives that the number S_n of possible schedules corresponds to the n^{th} small Schröder number, that is, the generating function for S_n is

$$S(t) = \frac{(1 - 3z - \sqrt{1 - 6z + z^2})}{4z}.$$



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