Algebraic succession rules and Lattice paths with an infinite set of jumps

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Abstract

Whereas walks on $\mathbb{N}$ with a finite set of jumps were the subject of numerous studies, walks with an infinite number of jumps remain quite rarely studied, at least from a combinatorial point of view. A reason is that even for relatively well structured models, the classical approach with context-free grammars fails as we deal with rewriting rules over an infinite alphabet. However, several classes of such walks offer a surprising structure: in this article, we show that one can make explicit the generating functions of the number of walks (with respect to their length) between two fixed points. We also give several theorems on their nature (rational, algebraic). In fact, we mostly deal with succession rules of the type

$$(k) \rightsquigarrow (0)^{e_k}(1)^{e_{k-1}} \dots (k-1)^{e_1}(k)^{e_0} \dots (k+a)^{e_{-a}},$$

for which we show that the associated generating function $F(z)$ is algebraic if the generating function $E(z)$ of the e_k's is rational (via a new combinatorial argument: a decomposition of the paths which leads to an algebraic equation satisfied by the noncommutative generating function). Via an analytical argument (the kernel method), we also show a stronger result: if $E(z)$ is algebraic, then $F(z)$ is algebraic. When $a = 1$, this leads to remarkably simple formulae which can also be proved with a Riordan array approach. This generalises all the previously known results.

We end with some examples of recent problems in combinatorics or theoretical computer science which lead to such rules.

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22 January 2003

1 Introduction

A considerable number of problems from computer science deals with a sum of independent identical distributed random variables $\Sigma_n = X_1 + X_2 + \ldots + X_n$ (where each of the X_i's assumes integer values). We consider here the following model of random walks: the walk starts (at time 0) from a point Σ_0 of $\mathbb{Z}$ and at time n, one makes a jump $X_n \in \mathbb{Z}$; so the new position is given by the recurrence $\Sigma_n = \Sigma_{n-1} + X_n$ where, when $\Sigma_{n-1} = k$, the jump X_n is constrained to belong to a fixed set $\mathcal{P}_k$ (that is, the possible jumps depend on the position of the walk).

These "walks on $\mathbb{Z}$" are homogeneous in time (that is to say, the set of jumps when one is at position k is independent from the time). When the positions Σ_n's are constrained to be nonnegative, we talk about "walks on $\mathbb{N}$". The probabilistic model under consideration here is the uniform distribution on all paths of length n.

When the sets $\mathcal{P}_k$'s are equal to a fixed set $\mathcal{P}$ (the simplest interesting case being $\mathcal{P} = \{-1, +1\}$), the corresponding walks have been deeply studied both in combinatorics (Dyck paths,...) and in probability theory (coin flipping, ...). We refer to [4] for enumerative and analytical studies of such "walks on $\mathbb{N}$ with a finite set of jumps". When the sets $\mathcal{P}_k$'s are unbounded, both enumeration and asymptotics become cumbersome: contrary to the previous case, the walks are not space-homogeneous (the set of available jumps depends on the position) and it is not possible to generate them by context-free grammars (which are classically defined for finite alphabet only). However, if the sets $\mathcal{P}_k$'s have a "combinatorial" shape, it is reasonable to hope that the generating function associated to the corresponding walk would have some nice properties. We show here that this hope is legitimate and we present several classes of such walks, for which we are able to give the nature of their generating function.

Our results have potential impacts on the theory of generating trees, the enumeration and generation of combinatorial objects (general classes of lattice paths, constrained permutations, ...) and on the study of rewriting rules on an infinite alphabet.

A definition of the generating function associated to the walk is given in Section 2 where we also present the generating tree and Riordan array viewpoints. In Section 3, we give several theorems related to the nature of the generating functions associated to some walks (which deeply generalise previously known results from [1–8,15,17–22]). Then, we give some asymptotic results. In Section 4, we give some examples of problems in which some of the new classes of walks that we study in this article appear.

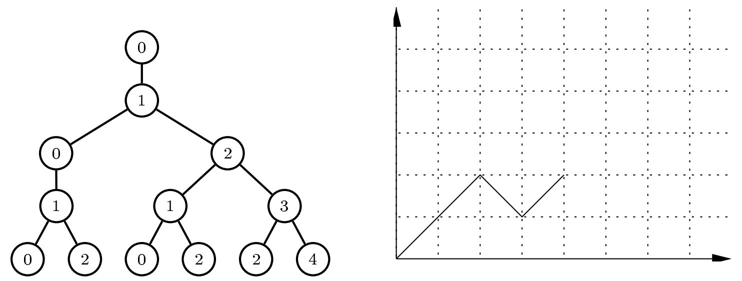

Fig. 1. The generating tree of the walk on $\mathbb{N}$ with jumps $\mathcal{P} = \{+1, -1\}$ starting in 0 (and up to length $n = 4$). Each branch corresponds to a path. The branch $(0, 1, 2, 1, 2)$ corresponds to the path drawn on the lattice.

2 Lattice paths, generating trees, succession rules and their generating functions

In combinatorics, it is classical to represent a particular walk as a path in a two dimensional lattice. Thus the drawing corresponds to the walk of length n linking the points $((0, \Sigma_0), (1, \Sigma_1), \ldots, (n, \Sigma_n))$. It is also convenient to represent all the walks of length $\leq n$ as a tree of height n, where the root (at level 0 by convention) is labelled with the starting point of the walks and where the label of each node at level n encodes a possible position of the walk (see Figure 1).

Let $f_{n,k}$ be the number of walks on $\mathbb{N}$ of length n going from the starting point to k (or, equivalently, the number of nodes with label k at level n in the tree). We want to find the bivariate generating function

$$F(z, u) = \sum_{n \geq 0} f_n(u) z^n = \sum_{k \in \mathbb{N}} F_k(z) u^k = \sum_{k \in \mathbb{N}, n \geq 0} f_{n,k} u^k z^n \,. \tag{1}$$

where u encodes the final altitude of the walk (the label in the tree), z the length of the walk (the level in the tree), and where $f_n(u)$ is a Laurent polynomial (that is, a polynomial with finitely many monomials of negative and positive degree).

2.1 Generating trees and succession rules

The concept of generating trees has been used from various points of view and was introduced in the literature by Chung, Graham, Hoggatt and Kleiman [10] to examine the reduced Baxter permutations.

We define here a generating tree as a rooted labelled tree with the property that if two nodes have the same label then, for any integer ℓ, they have exactly

the same number of children with label ℓ. For readability, we often write the labels in parentheses. Thus, a generating tree is fully defined by:
1) the label of the root (that we also call "axiom");
2) a set of rules $\{(k) \rightsquigarrow \mathcal{M}_k\}_{k\in\mathbb{N}}$ explaining how to derive from the label of a parent the labels of its children. ($\mathcal{M}_k$ is a multiset [1] of labels.)

Point 2) defines what we call a *succession rule.* The multisets $\mathcal{M}_k$ are directly related to the multisets $\mathcal{P}_k$ (the allowed jumps introduced in Section 1) via the relation $\mathcal{M}_k := \{k + x, x \in \mathcal{P}_k\}$. For example, Figure 1 illustrates the upper part of the generating tree which corresponds to the set of rules $\{(k) \rightsquigarrow (k-1)(k+1)\}_{k\in\mathbb{N}}$ with 0 as label of the root. That is, one has in this case $\mathcal{P}_k = \{-1, +1\}$ and $\mathcal{M}_k = \{k-1, k+1\}$. In what follows, instead of writing

$$((0), \{(k) \rightsquigarrow (k-1)(k+1)\}_{k\in\mathbb{N}}) ,$$

we use the more readable notation

$$[(0),\ (k) \rightsquigarrow (k-1)(k+1)\,] ,$$

or alternatively

$$\begin{cases} (0) \\ (k) \rightsquigarrow (k-1)(k+1)\,. \end{cases}$$

Note that we only consider nonnegative walks, thus when a rule gives a negative label, we simply ignore this label. In the above case, when $k = 0$ the rule is thus $(0) \rightsquigarrow (1)$ and not $(0) \rightsquigarrow (-1)(1)$. If a label is repeated, we directly write $(k)^n$ instead of $(k)\dots(k)$ (n occurrences). This corresponds to walks with multiplicities, or if one wants, to distinguish two occurrences of the same label in a succession rule by colouring them in two different colours.

The method of generating trees was also successfully used by West [25], Dulucq, Gire, and Guibert [12–14], for the enumeration of permutations with forbidden sequences (see Fig. 2). In fact, the kind of rewriting rules under consideration here were intensively studied partly because they are useful to solve some cases of the following famous conjecture:

Conjecture 1 (Stanley–Wilf) *For any given pattern, there exists a constant C such that there are asymptotically $O(C^n)$ permutations of length n avoiding this pattern.*

[1] Multisets are sets in which repetitions are allowed. E.g., for multisets, one has $\{1, 1, 2\} \cup \{1, 2\} = \{1, 1, 1, 2, 2\}$.

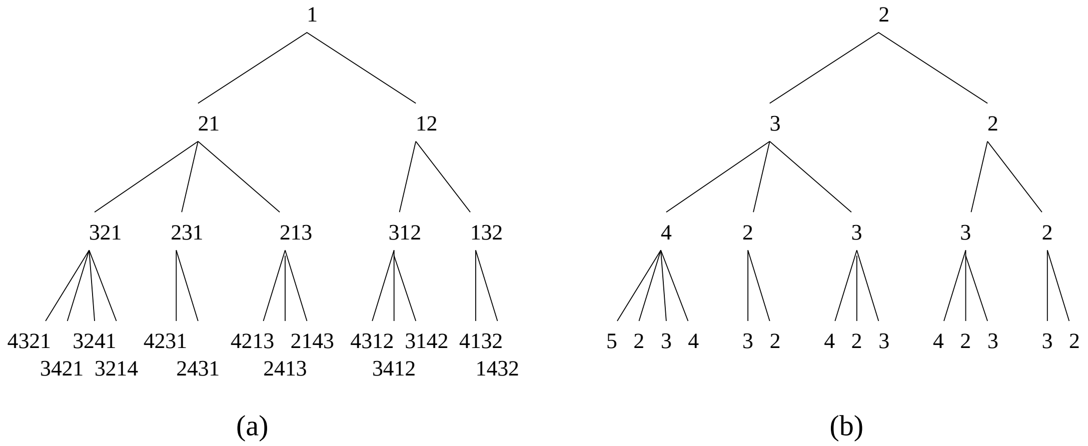

Fig. 2. The generating tree of 123-avoiding permutations. (a) Nodes labelled by the permutations. (b) Nodes labelled by the numbers of children. It can be proved that the right tree corresponds to the rule $[(2),(k) \rightsquigarrow (2)\dots(k+1)]$.

This conjecture shows that to forbid a pattern is a strong constraint (permutations with a forbidden pattern are of density zero in the whole set of permutations). For any fixed pattern, the algebraicity of the generating function of permutations avoiding this pattern would be a proof of this conjecture. However, it is not possible to solve all the cases by this approach (as some patterns lead to non-D-finite [2] generating functions).

These last years, the concept of generating tree has been intensively exploited by Barcucci, Del Lungo, Ferrari, Pergola, Pinzani, and Rinaldi [6,7,17,18] in relation with the ECO method (ECO stands for enumeration of combinatorial objects) which allows the enumeration and recursive construction of various classes of combinatorial objects. In fact, the succession rule approach has several equivalent interpretations, ECO systems, discrete random walks, infinite automata or Riordan arrays (see later). For all these problems, it is interesting to classify the rules according to the nature (rational, algebraic, transcendental) of the corresponding generating function $F(z,u)$. This program has been proposed by Pinzani and *al.* [6,7,17,18] in the area of ECO systems (the so called "ECO systems" are the generating trees where each integer has exactly k successors). A classical and easy result is that finite succession rules have rational generating function since they correspond to a regular language. Another result (proved in [3]) is that every finite transformation of the succession rule

$$(k) \rightsquigarrow (1)(2)\dots(k)(k+1)$$

leads to an algebraic generating function. In the same paper are also described succession rules leading to exponential generating functions having a

[2] A series $F(z)$ is said to be *holonomic*, or *D-finite*, if it satisfies a linear differential equation with polynomial coefficients in z. Equivalently, its coefficients f_n satisfy a linear recurrence relation with polynomial coefficients in n.

nice closed-form formula which have been more extensively studied by Corteel in [11]. Our paper is principally devoted to the study of succession rules having algebraic generating function.

In a first step, our approach is closely related with Schützenberger's methodology, which consists in finding first a bijection between the objects and the words of an algebraic language and then a non ambiguous grammar for the language. Taking the commutative image leads to an algebraic system for the generating function. For a succession rule, we define its noncommutative formal power series using the infinite alphabet of positive integers. We use a new operation $\oplus$ which allows us to get a non ambiguous decomposition of the formal power series associated to the generating tree. We deduce algebraic equation by taking the commutative image of the formal power series. This method allows us to get an algebraic decomposition of the general succession rule

$$(k) \rightsquigarrow (1)\dots(k-1)(k)^{e_0}\dots(k+a)^{e_{-a}},$$

for any finite sequence (e_i), and more generally for the succession rule

$$(k) \rightsquigarrow (1)^{e_{k-1}}\dots(k-1)^{e_1}(k)^{e_0}\dots(k+a)^{e_{-a}},$$

for any sequence $(e_i)_{i=-a}^{\infty}$ proving thereby that the generating function of the generating tree is *algebraic* when the sequence (e_i) is *rational*. This gives a combinatorial proof for a generalisation of the results of [3].

In a second step, we give some analytical proofs (based on the kernel method) of the algebraicity of the generating function associated to the generating tree when the sequence (e_i) in the succession rule is *algebraic*.

2.2 Noncommutative generating functions for succession rules

It is convenient to see a generating tree (defined in the previous subsection) as the infinite tree constructed with a root labelled by the axiom and where each node labelled k has sons labelled according to the succession rule.

For a generating tree $\mathcal{T}$, we define the language $\mathcal{L}$ as the set of words over $\mathbb{N}$, beginning by the axiom r and in which each letter (k) is followed by a letter (if any) which belongs to the multiset $\mathcal{M}_k$. Each word w of $\mathcal{L}$ corresponds to at least one branch [3] of $\mathcal{T}$. For each word $w \in \mathcal{L}$, let $m(w)$ be the number of branches in the generating tree $\mathcal{T}$ corresponding to the word w. We denote by S the noncommutative formal power series $S = \sum_{w\in\mathcal{L}} m(w)w$.

[3] By branch of the infinite tree $\mathcal{T}$, we mean any sequence of labels corresponding to a branch of any finite subtree of $\mathcal{T}$. Figure 1 gives an example.

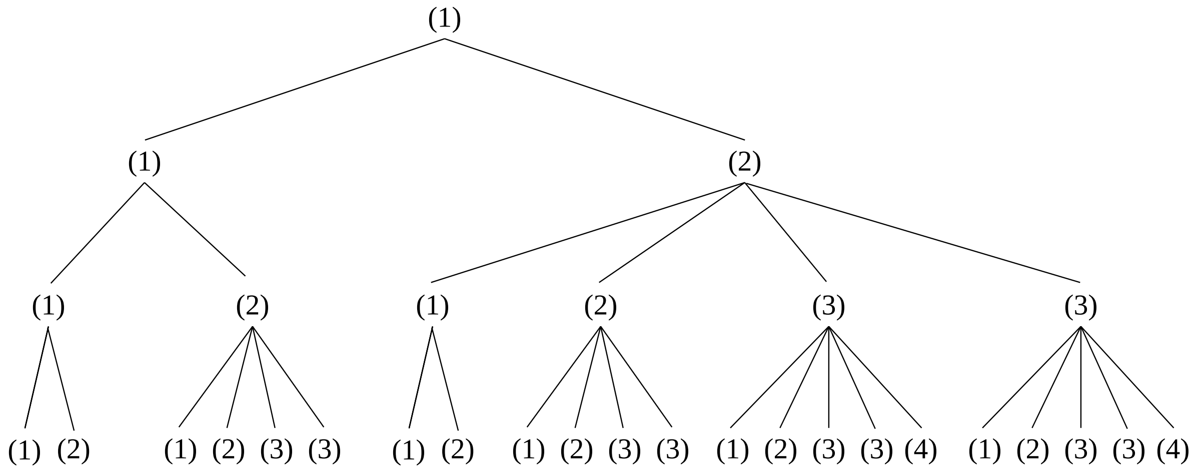

Fig. 3. Truncated generating tree of $[(1), (k) \rightsquigarrow (1)(2)(3)(3)(4)\dots(k)(k+1)]$. The associated generating function is $F(z,1) = z + 2z^2 + 6z^3 + 22z^4 + \dots$ and the corresponding noncommutative GF is $S = 1 + 11 + 12 + 111 + 112 + 121 + 122 + 2.123 + \dots$

By construction, the generating tree $\mathcal{T}$ and the noncommutative formal power series S have the same generating function

$$F(z,1) = \sum_{n\in\mathbb{N}} f_n(1)z^n = \sum_{n\in\mathbb{N}} \left(\sum_{w\in\mathcal{L}, |w|=n+1} m(w) \right) z^n .$$

We use standard external product and concatenation over the noncommutative formal power series: For any real x and for any word v, one has

$$xS := \sum_{w\in\mathcal{L}} (xm(w))w \qquad \text{and} \qquad v.S := \sum_{w\in\mathcal{L}} m(w)(v.w).$$

We now define the "shift" operation (that we write $\oplus$) as follows:

Definition 2 *For $i \in \mathbb{N}$, we define $i^{\oplus} := i + 1$. By extension if $w = w_1 \dots w_n$ is a word with n letters, then $w^{\oplus} := w_1^{\oplus} \dots w_n^{\oplus}$ and $S^{\oplus} := \sum_{w\in\mathcal{L}} m(w)w^{\oplus}$.*

Clearly, the generating functions associated to S and $S^{\oplus}$ are equal.

2.3 Riordan arrays

We introduce now the concept of *matrix associated to a generating tree*: this is an infinite matrix $\{d_{n,k}\}_{n,k\in\mathbb{N}}$ where $d_{n,k}$ is the number of nodes at level n with label $k + r$, where r is the label of the root. For example, the matrix associated to the generating tree of the Figure 1 (walk with jumps $+1, -1$) is the following:

$n\backslash k$	0	1	2	3	4
0	1				
1	0	1			
2	1	0	1		
3	0	2	0	1	
4	2	0	3	0	1

Many such matrices can be studied from a *Riordan array* viewpoint. In fact, the concept of a Riordan array provides a remarkable characterisation of many lower triangular arrays that arise in combinatorics and algorithm analysis. The theory has been introduced in the literature in 1991 by Shapiro, Getu, Woan, and Woodson [23]. Riordan arrays are a powerful tool in the study of many counting problems having a flavour of Lagrange inversion [19].

A Riordan array is an infinite lower triangular array $\{d_{n,k}\}_{n,k\in\mathbb{N}}$, defined by a pair of formal power series $(d(z), h(z))$, such that the k-th column is given by $d(z)(zh(z))^k$, i.e.:

$$d_{n,k} = [z^n]d(z)(zh(z))^k, \qquad n, k \geq 0.$$

From this definition, one has $d_{n,k} = 0$ for $k > n$. The bivariate generating function for the Riordan array is:

$$\sum_{n,k\geq 0} d_{n,k}u^k z^n = \frac{d(z)}{1 - uzh(z)}.$$

In what follows, we always assume that $d(0) \neq 0$; if we also have $h(0) \neq 0$ then the Riordan array is said to be *proper*; in the proper-case the diagonal elements $d_{n,n}$ are different from zero for all $n \in \mathbb{N}$. The most simple example is the Pascal triangle for which one has

$$\binom{n}{k} = [z^n]\frac{1}{1-z}\left(\frac{z}{1-z}\right)^k,$$

where we recognise the proper Riordan array with $d(z) = h(z) = 1/(1 - z)$. Proper Riordan arrays are characterised by the existence of a sequence $A = (a_i)_{i\in\mathbb{N}}$ with $a_0 \neq 0$, called the A-*sequence*, such that every element $d_{n+1,k+1}$ can be expressed as a linear combination, with coefficients in A, of the elements in the preceding row, starting from the preceding column:

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \dots$$

It can be proved that $h(z) = A(zh(z))$, $A(z)$ being the generating function for the sequence A. For example, for the Pascal triangle one has: $A(z) = 1+z$ and the previous relation reduces to the well-known recurrence relation for binomial coefficients. The A-sequence doesn't characterise completely $(d(z), h(z))$ because $d(z)$ is independent of $A(z)$. But it can be proved that there exists a unique sequence $Z = (z_0, z_1, z_2, \ldots)$, such that every element in column 0 can be expressed as a linear combination of all the elements of the preceding row:

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \ldots$$

This property has been recently studied in [19], where it is proved that $d(z) = d(0)/(1 - zZ(zh(z)))$, $Z(z)$ being the generating function for Z. Thus the triple $(d(0), Z(z), A(z))$ characterises every proper Riordan array. We use these claims in Theorem 10.

3 Generating functions of succession rules

This section contains the main results of our article. We give several theorems, making explicit the generating functions associated to different kind of rules ("rational" exponents: Theorem 4, "algebraic" exponents: Theorem 9, ...).

3.1 Lattice paths and generating trees

Consider a function $e(k, i)$ which is going from $\mathbb{N}^2$ to $\mathbb{N}$. We now fix an integer $a > 0$ (a corresponds to the largest positive possible jump; so we restrict here our attention to functions such that $e(k, i) = 0$ for any k as soon as $i > a$). Then the walks with an infinite set of jumps under consideration here are of the following kind:

$$\begin{cases} (r) \\ (k) \rightsquigarrow (0)^{e(k,0)}(1)^{e(k,1)} \ldots (k-1)^{e(k,k-1)}(k)^{e(k,k)} \ldots (k+a)^{e(k,k+a)} \end{cases} \tag{2}$$

where the exponent $e(k, i)$ is the multiplicity of the jumps from k to i and where r is the starting position of the walk (or equivalently, the root of the associated generating tree). In what follows, we often (but not always) consider the case for which $e(k, i) = e_{k-i}$ (where $(e_k)_{k \in \mathbb{Z}}$ is a fixed sequence).

If the sequence $(e(k, i))_k$ (for a fixed i) is ultimately 0, then the situation covers the case of walks with a finite set of jumps [4]. If the sequence is ultimately 1, then this covers the case of "factorial rules" which are of great interests for the generation of combinatorial objects [8] and for which it was proved in [3] that the associated generating functions are algebraic.

We still note $f_{n,k}$ the number of walks on $\mathbb{N}$ of length n going from the starting point to k and we want to find the bivariate generating function $F(z,u) = \sum_{n,k\geq 0} f_{n,k} u^k z^n$. These random walks on $\mathbb{N}$ can equivalently be seen as lattice paths, generating trees, and also as Riordan arrays (when $a=1$).

In Tables 1 and 2 (see at the end of this article), we give a list of succession rules with simple combinatorial patterns, the reference to famous numbers or combinatorial problems they refer to, the generating function $F(z,1)$, and the numbers identifying the corresponding sequences in the *On-Line Encyclopedia of Integer Sequences* `http://www.research.att.com/~njas/sequences/`; ECS stands for the *Encyclopedia of Combinatorial Structures*, a database reachable via `http://algo.inria.fr/encyclopedia/`.

3.2 Succession rules: "rational" exponents

In this section, we study succession rules having the following general form

$$[(1), (k) \rightsquigarrow (1)^{e_{k-1}} \dots (k-1)^{e_1} (k)^{e_0} \dots (k+a)^{e_{-a}}] \,.$$

"Rational" exponents means here that the generating function $E(z)$ of the e_k's (which are nonnegative integers) is rational.

Using decomposition of paths, we prove the algebraicity of the associated generating function $F(z,1)$, first when the sequence $(e_i)_{i>0}$ is constant equal to one (Theorem 3), and then when the sequence (e_i) follows a linear recurrence, that is when the e_i's are coefficients of a rational generating function (Theorem 4).

Theorem 3 *The noncommutative generating function S associated to the generating tree*

$$[(1), (k) \rightsquigarrow (1) \dots (k-1)(k)^{e_0} \dots (k+a)^{e_{-a}}]$$

satisfies the following equation

$$S = (1) + (1) \sum_{i=0}^{a} e_{-i} S^{i\oplus} \prod_{j=0}^{i-1} (\epsilon + S^{j\oplus}),$$

where $S^{i\oplus} = (S^{(i-1)\oplus})^{\oplus}$ and $S^{0\oplus} = S$.

Consequently, the (commutative) generating function $F(z,1)$ of the generating *tree is algebraic and satisfies*

$$F(z,1) = z + zF(z,1)\sum_{i=0}^{a} e_{-i}(1+F(z,1))^i \,.$$

Example: For the generating tree associated to $[(1), (k) \rightsquigarrow (1)\dots(k+1)]$, this gives $S = (1) + S + S^{\oplus}(\epsilon + S)$.

Remark: The algebraicity of $F(z,1)$ for such generating trees was first proved analytically in [3] (via a proof which is leading to a neat closed-form formula). We give here a *combinatorial* proof of this algebraicity (via a neat decomposition of the tree).

Proof. The proof is deduced from the recursive decomposition of the paths in the generating tree. We need to define $({}_rS)$ as the formal sum of the paths in the generating tree obtained by replacing the axiom by r:

$$[(r), (k) \rightsquigarrow (1)\dots(k-1)(k)^{e_0}\dots(k+a)^{e_{-a}}] \,.$$

We can write recursively $({}_rS)$ using the following non ambiguous decomposition (see Fig. 4). Let $w \neq r$ be a non trivial path of $({}_rS)$, then w can be written $w = r.u$

- if each letter of u is $\geq r$ then $u = v^{(r-1)\oplus}$ where v is a path of the generating tree,
- if not, let m the first letter $< r$ in u, so u can be written $v_1^{(r-1)\oplus}v_2$ where v_1 is a path of the generating tree and v_2 is a path of $({}_mS)$, v_2 being the longest suffix of u beginning by m.

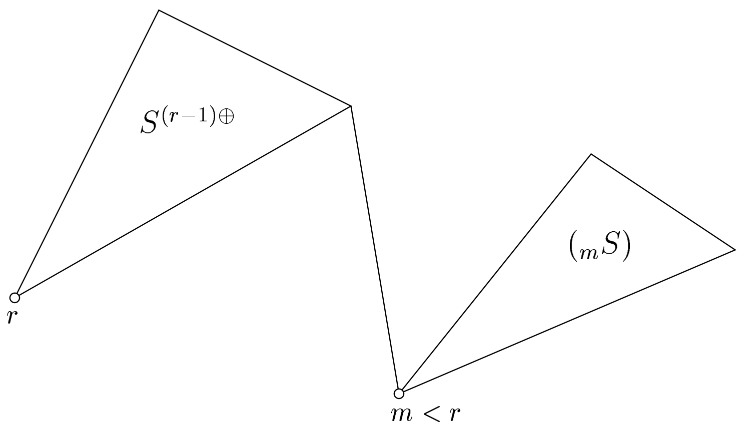

Fig. 4. Decomposition of $({}_rS)$.

One has $({}_rS) = S^{(r-1)\oplus}(\epsilon + \sum_{m=1}^{r-1} ({}_mS))$. It is easy to see that

$$({}_{r+1}S) = S^{r\oplus} \prod_{m=0}^{r-1} (\epsilon + S^{m\oplus}) \,.$$

The equality $S = (1) + (1) \sum_{r=0}^{a} e_{-r}({}_{r+1}S)$ concludes the proof. □

The algebraic equation satisfied by the algebraic generating function given in the small catalogue of ECO-systems of [3] can be deduced from the previous theorem. For instance, this is the case of Motzkin numbers, Schröder numbers and ternary trees. For the general case, that we consider now, the difficulty is to deal with the e_i jumps from (k) to $(k-i)$.

Theorem 4 *Consider $E(z) = \sum_{i \geq -a} e_i z^i$. The rationality of $E(z)$ implies the algebraicity of the generating function $F(z,1)$ of the generating tree*

$$[(1), (k) \rightsquigarrow (1)^{e_{k-1}} \ldots (k-1)^{e_1} (k)^{e_0} \ldots (k+a)^{e_{-a}}] \,.$$

Proof. We begin by giving the different equations obtained from the recursive decomposition of the paths in the generating tree. As in the proof of Theorem 3, we need to define $({}_rS_i)$ as the formal sum of the paths ending by i in the following generating tree

$$[(r), (k) \rightsquigarrow (1)^{e_{k-1}} \ldots (k-1)^{e_1} (k)^{e_0} \ldots (k+a)^{e_{-a}}] \,.$$

We write (S_i) for $({}_1S_i)$. Applying the same non ambiguous decomposition as in Theorem 3 and considering the last letter of each factor (see Fig. 5), we get

$$({}_{r+1}S_i) = (S_{i-r})^{r\oplus} + \sum_{m=1}^{r} \sum_{j \geq 1} e_{j+r-m} (S_j)^{r\oplus} ({}_mS_i). \tag{3}$$

Let ${}_rF_i(z)$ be the generating functions of $({}_rS_i)$ (paths beginning in r and ending in i). By convention, ${}_rF_i = 0$ for $i \leq 0$ or $r \leq 0$. One has $F_i = ({}_1F_i)$ (as 1 is the root of the generating tree). Let $G_i := \sum_{j \geq 1} e_{i+j-1} F_j$ for $1 \leq i \leq p$ and $H_n(z) := \sum_{n=i_1+\ldots+i_h} G_{i_1}(z) \ldots G_{i_h}(z)$ for $n \geq 0$ with the convention that $H_0(z) := 1$. Note that H_n is a polynomial in $G_1, \ldots, G_a$. From Equation (3), one gets

$$\begin{aligned} {}_{r+1}F_i(z) &= F_{i-r}(z) + \sum_{m=1}^{r} \sum_{j \geq 1} e_{j+r-m} \, F_j(z) \; ({}_mF_i(z)) \\ &= F_{i-r}(z) + \sum_{m=1}^{r} G_{r-m+1}(z) \; ({}_mF_i(z)) \,. \end{aligned}$$

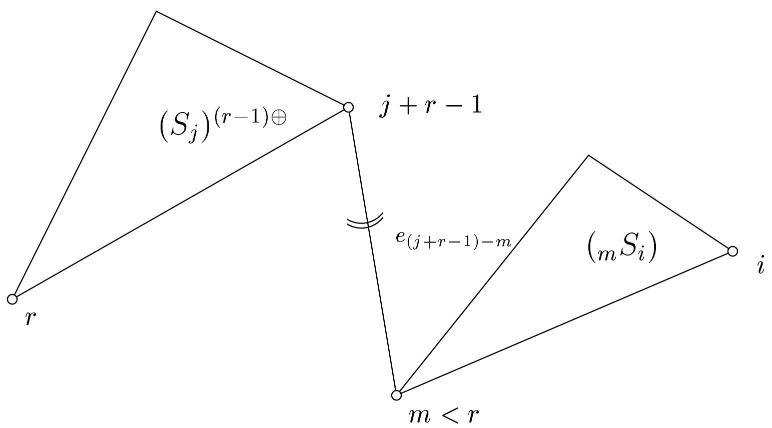

Fig. 5. Decomposition of $({}_rS_i)$.

For $k \geq 2$, decomposing F_k according to the first positive jump, gives

$$F_k = z\sum_{m=0}^{a} e_{-m}({}_{m+1}F_k).$$

Using the fact that

$$\begin{aligned}({}_{r+1}F_i) &= F_{i-r} + \sum_{m=1}^{r} G_{r-m+1}\sum_{j=0}^{m-1} H_{m-1-j}F_{i-j}\\ &= F_{i-r} + \sum_{m=0}^{r-1} F_{i-m}\sum_{j=0}^{r-m-1} G_{r-m-j}H_j\\ &= F_{i-r} + \sum_{m=0}^{r-1} F_{i-m}H_{r-m}\\ &= \sum_{m=0}^{r} H_{r-m}F_{i-m}\,,\end{aligned}$$

one has

$$\begin{aligned}F_k &= z\sum_{m=0}^{a} e_{-m}\sum_{i=0}^{m} H_{m-i}F_{k-i}\\ &= z\sum_{j=0}^{a} F_{k-j}\sum_{i=j}^{a} e_{-i}H_{i-j}\\ &= zF_k\sum_{i=0}^{a} e_{-i}H_i + z\sum_{j=1}^{a} F_{k-j}\sum_{i=j}^{a} e_{-i}H_{i-j}.\end{aligned}$$

For $k=1$, one gets $F_1 = z + z\sum_{i=0}^{a} e_{-i}({}_{i+1}F_1)$. Let $b_j = \sum_{i=j}^{a} e_{-i}H_{i-j}$, one has

$$\begin{cases} F_1 &= \frac{z}{1-zb_0} \\ F_k &= \frac{z}{1-zb_0}\sum_{i=1}^{a} zb_i\, F_{k-i}\,, \qquad \text{for} \quad k>1. \end{cases}$$

Let M be the following a by a matrix (whose entries are rational functions in $G_1,\ldots,G_a$),

$$M := \begin{pmatrix} \frac{zb_1}{1-zb_0} & \frac{zb_2}{1-zb_0} & \ldots & \frac{zb_{a-1}}{1-zb_0} & \frac{zb_a}{1-zb_0} \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} F_k \\ F_{k-1} \\ \vdots \\ F_{k-a+1} \end{pmatrix} = M \begin{pmatrix} F_{k-1} \\ F_{k-2} \\ \vdots \\ F_{k-a} \end{pmatrix} = M^{k-1} \begin{pmatrix} F_1 = \frac{z}{1-zb_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Before to go on, one needs the following lemma.

Lemma 5 *The rationality of the sequence (e_i) implies the algebraicity of the sequence (G_i).*

Proof. If the sequence $(e_i)_{i\geq -a}$ is rational, then the sequence $(e_k)_{k\geq 1}$ is also rational and there exist two polynomials P and Q such that $\sum_{k\geq 1} e_k z^{k-1} = \frac{P(z)}{Q(z)}$, with $Q(0)\neq 0$. Thus one has $\sum_{k\geq 1} e_k M^{k-1} = P(M)Q(M)^{-1}$, because $Q(M)$ is invertible. Indeed, decomposing $Q(z)$ in $\mathbb{C}$ leads to $Q(z) = c\prod_{i=1}^{deg(Q)}(z-\rho_i)$, so that $\mathrm{Det}(Q(M)) = c\prod_{i=1}^{deg(Q)}\mathrm{Det}(M-\rho_i I)$, which is obviously nonzero by computing,

$$\mathrm{Det}(M-\rho I) = (-1)^{a+1}(-\rho^a + \frac{z}{1-zb_0}\sum_{m=1}^{a} b_m\rho^{a-m}).$$

Thus we can write an algebraic system of a equations for $G_1, \ldots, G_a$,

$$\begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_a \end{pmatrix} = \sum_k e_k \begin{pmatrix} F_k \\ F_{k-1} \\ \vdots \\ F_{k-(a-1)} \end{pmatrix} = \sum_k e_k M^{k-1} \begin{pmatrix} \frac{z}{1-zb_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{P(M)}{Q(M)} \begin{pmatrix} \frac{z}{1-zb_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The Jacobian of this system is equal to the identity for $z = 0$ so the G_i's are algebraic functions of z. $\square$

Moreover F_k is algebraic for all $k \geq 1$:

$$\begin{pmatrix} F_k \\ F_{k-1} \\ \vdots \\ F_{k-a+1} \end{pmatrix} = M^{k-1} \begin{pmatrix} \frac{z}{1-zb_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Finally, this leads to

$$\textstyle\sum_{k\geq 1} \begin{pmatrix} F_k \\ F_{k-1} \\ \vdots \\ F_{k-a+1} \end{pmatrix} = (M-1)^{-1} \begin{pmatrix} \frac{z}{1-zb_0} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

taking the first entry gives that $F(z,1) = \sum_{k\geq 1} F_k(z)$ is algebraic. $\square$

Remark: In fact Lemma 5 can be extended. Indeed, if $E(z)$ is algebraic, then the G_i's are still algebraic. For this, let P the bivariate polynomial such that $P(E,z) = 0$. Now, consider the first and the third member in last formula in the proof of the Lemma. Multiplying them by adequate monomials $E(M)^i M^j$ and summing over adequate values of (i,j) allows to get $P(E(M), M)$ in the third member. As this is equal to 0, one thus gets a system of a equations for the G_i's (with algebraic coefficients). The rest of the proof still implies the algebraicity of $F(z,1)$. We don't push the proof in this direction because, in Theorem 9 hereafter, we give another proof which leads to a neat closed-form formula for $F(z,u)$.

Theorem 4 above allows us to generalise a result from [3] concerning finite transformations of $(k) \rightsquigarrow (1)\dots(k-1)(k)(k+1)$. A finite transformation of a rule consists in adding a fixed integer to one (resp. all) succession rule(s). The noncommutative formal power series approach allows us to interpret finite transformations and show that they do not change the algebraicity of the generating function. Moreover, the property of algebraicity does not depend on the choice of the axiom.

Theorem 6 *Consider $E(z) = \sum_{i \geq -a} e_i z^i$. If $E(z)$ is algebraic, then all "finite transformations" (as defined above) of the succession rule*

$$(k) \rightsquigarrow (1)^{e_{k-1}} \dots (k-1)^{e_1} (k)^{e_0} \dots (k+a)^{e_{-a}}$$

lead to an algebraic associated generating function $F(z,1)$. More generally, all finite transformations of the succession rule $(k) \rightsquigarrow \mathcal{M}_k$ lead to an algebraic associated generating function $F(z,u)$ as soon as the original bivariate generating function is algebraic.

Proof. For any fixed nonnegative integer c, let $\mathcal{T}$, $\mathcal{T}'$, and $\mathcal{T}''$ be the following the generating trees:

$$\mathcal{T} = \begin{cases} (r) \\ (k) \rightsquigarrow \mathcal{M}_k \,, \end{cases}$$

$$\mathcal{T}' = \begin{cases} (r) \\ (k) \rightsquigarrow \mathcal{M}_k \cup (c) \,, \end{cases}$$

$$\mathcal{T}'' = \begin{cases} (r) \\ (k_0) \rightsquigarrow \mathcal{M}_{k_0} \cup (c) \\ (k) \rightsquigarrow \mathcal{M}_k \,, \qquad \text{for} \quad k \neq k_0 \,. \end{cases}$$

Thus, $\mathcal{T}'$ and $\mathcal{T}''$ are finite transformations of $\mathcal{T}$. Let S, S', and S'' (resp. F, F', and F'') be the formal sum of paths (resp. the commutative generating functions) associated to $\mathcal{T}, \mathcal{T}'$, and $\mathcal{T}''$. As in the proofs of Theorem 3 and Theorem 4, let (S_k) be the formal sum of paths ending by k and $({}_cS)$ be the formal sum of the paths in the generating tree $[(c), (k) \rightsquigarrow (1)^{e_{k-1}} \dots (k-1)^{e_1} (k)^{e_0} \dots (k+a)^{e_{-a}}]$, that is the original generating tree $\mathcal{T}$ where the axiom r has been replaced by c.

As $S'' = S + S_{k_0}.({}_cS_{k_0})^*({}_cS)$, this gives $F''(z,u) = F(z,u) + \dfrac{F_{k_0}(z) \quad {}_cF(z,u)}{1 - {}_cF_{k_0}(z)}$ where the right member involves only functions which are known to be algebraic, thus F'' is also algebraic. Similarly, the relation $S' = S.({}_cS)^*$, gives the algebraicity of F'. □

By duality, similar results hold if you *remove* a label from one (or all) rule(s). Note also that there is no difficulty to apply the same kind of proofs to other transformations like $[(r), (k_0) \rightsquigarrow \mathcal{M}_{k_0}, (k) \rightsquigarrow \mathcal{M}_k \cup (c)]$.

3.4 Succession rules: polynomial exponents and no negative bounded jump allowed

Theorem 7 *For any constant $B \geq 0$, the generating tree*

$$[(r), (k) \rightsquigarrow (0)^{e(k,0)} \ldots (B)^{e(k,B)} \quad (k)^{e_0} \ldots (k+a)^{e_{-a}}]$$

(where $e(k,0), \ldots, e(k,B)$ are polynomial in k, $e(k,i) = 0$ for $B < i < k$ and $e(k,i) = e_{k-i}$, some fixed constants, for $i \geq k$) has a rational generating function $F(z,u)$.

Proof. First, we illustrate the general case by the following example:

$$\begin{cases} (0) \\ (k) \rightsquigarrow (0)^{k^2}(2)^{3k-1}(3)(k)(k+1)^2(k+3)^5\,, \end{cases} \tag{4}$$

for which $B = 3$, the polynomials in k are $e(k,0) = k^2, e(k,1) = 0, e(k,2) = 3k-1, e(k,3) = 1$, and the fixed constants are $e_0 = 1$, $e_{-1} = 2$, $e_{-2} = 0$, $e_{-3} = 5$.

The part $(k) \rightsquigarrow (0)^{k^2}$ implies a transformation $u^k \rightsquigarrow k^2u^0$. The part $(k) \rightsquigarrow (2)^{3k-1}$ implies a transformation $u^k \rightsquigarrow (3k-1)u^2$. The part $(k) \rightsquigarrow (3)$ implies a transformation $u^k \rightsquigarrow u^3$. It is possible to perform all these transformations using the derivation[4], evaluation in $u = 1$ and multiplication by a monomial: in the first case, the multiplicity k^2 is obtained by $\partial(u\partial(u^k))$ and then evaluating in $u = 1$; for the second case, the multiplicity $3k-1$ is obtained by taking $\partial(u^{3k})/u$ and then evaluating in $u = 1$; for the third case simply evaluate in $u = 1$ and multiply by u^3. The part $(k) \rightsquigarrow (k)(k+1)^2(k+3)^5$ gives $u^k \rightsquigarrow P(u)u^k$ where $P(u) = 1 + 2u + 5u^3$. All these transformations are in fact linear, so to act on u^k or a polynomial in u (like $f_n(u)$) is the same.

[4] We denote the derivation with respect to u by ∂_u or by ∂ or $'$ when there is no ambiguity. We also write abusively $\partial_uF(z,1)$ for $(\partial_uF)(z,1)$.

Finally, evaluating $\partial(u\partial f_n(u))$ in $u=1$ gives $f_n''(1)+f_n'(1)$ and evaluating $u^2\partial_u f_n(u^3)/u$ in $u=1$ gives $u^2(3f_n'(1)-f_n(1))$, so these trivial simplifications gives the following recurrence:

$$f_{n+1}(u)=P(u)f_n(u)+u^0(f_n''(1)+f_n'(1))+u^2(3f_n'(1)-f_n(1))+u^3f_n(1)\,.$$

Multiplying by z^{n+1} and summing for $n\geq 0$ leads to the functional equation

$$(1-zP(u))F(z,u)=1+z(u^3-1)F(z,1)+z(3u^2+1)\partial_uF(z,1)+z\partial_u^2F(z,1)\,.$$

Taking the first 2 derivatives and instantiating in $u=1$ gives a rational system of full rank, hence $F(z,u)$ is rational:

$$F(z,u)=\frac{u^3(22z^2-112z^3-z)+u^2(480z^3-60z^2)+528z^3-250z^2+31z-1}{(1-zP(u))(872z^3-212z^2+30z-1)}\,.$$

For the general case, one has the following functional equation

$$(1-zP(u))F(z,u)=u^r+z\sum_{i=0}^{d}t_i(u)\partial_u^iF(z,1)$$

(d is the largest degree of the polynomials $e(k,i)$, and the t_i's are some Laurent polynomials which can be made explicit). Taking the first d derivatives and instantiating in $u=1$ gives a system (for $m=0,\dots,d$):

$$\partial_u^m u^r+\left(\sum_{i=0}^{m-1}\left(z\partial_u^m t_i(1)+z\binom{m}{i}\partial_u^{m-i}P(1)\right)\partial_u^iF(z,1)\right)$$
$$+\left(z\partial_u^m t_i(1)-(1-zP(1))\right)\partial_u^mF(z,1)+z\sum_{i=m+1}^{d}\partial_u^m t_i(1)\partial_u^iF(z,1)=0\,.$$

This gives a matricial equation $M.\overrightarrow{F}=\overrightarrow{v}$ where $\overrightarrow{v}=(u^r,0,\dots,0)^T$ and $\overrightarrow{F}=(\partial_u^0F(z,1),\dots,\partial_u^dF(z,1))^T$. The coefficients of the main diagonal of M are $-1+z\dots$ (as they are the coefficients of the $\partial_u^mF(z,1)$ summand) and all the other coefficient of M are monomials in z of degree 1. Thus, one has $[z^0]\det M=\pm 1$ and then $\det M\neq 0$. Consequently, this system is of full rank. Solving it gives rational expressions for the $\partial_u^iF(z,1)$ and for $F(z,u)$. $\square$

3.5 Succession rules: polynomial exponents and negative jumps allowed

We now give a generalisation of a result of [3] which was giving the algebraicity of "factorial rules": we allow here initial multiplicities which are not space-homogeneous.

Theorem 8 *For any constant $B \geq 0$, the generating tree*

$$[(r),(k) \rightsquigarrow (0)^{e(k,0)} \dots (B)^{e(k,B)}(B+1)\dots(k-b-1)(k-b)^{e_b}\dots(k+a)^{e_{-a}}]$$

(where $e(k,0),\dots,e(k,B)$ are polynomial in k, $e(k,i)=1$ for $B<i<k-b$ and $e(k,i)=e_{k-i}$, some fixed constants, for $i \geq k-b$) has an algebraic generating function $F(z,u)$.

Proof. We illustrate the general case by the following example:

$$[(0),(k) \rightsquigarrow (0)^{k^2}(2)^{3k^5-2}(6)(7)\dots(k-5)(k-4)^2(k-2)^3(k)(k+3)^2(k+23)]\,,$$

for which $B=5, b=4, a=23$, the polynomials in k are $e(k,0)=k^2$, $e(k,2)=3k^5-2$, $e(k,1)=e(k,3)=e(k,4)=e(k,5)=0$ and the fixed constants are $e_4=2$, $e_2=3$, $e_0=1$, $e_{-3}=2$, $e_{-23}=1$. One sets $P(u)=2u^{-4}+3u^{-2}+1+2u^3+u^{23}$, the recurrence is

$$f_{n+1}(u) = P(u)f_n(u) - \{u^{<0}\}P(u)f_n(u) + \sum_{i=0}^{5} t_i(u)\partial_u^i f_n(1)\,,$$

where $\{u^{<0}\}$ stands for the sum of the monomials in u with a negative degree. Multiplying by z^{n+1} and summing for $n \geq 0$ leads to the functional equation

$$(1-zP(u))F(z,u) = 1 - z\sum_{k=0}^{4-1} r_k(u)F_k(z) + z\sum_{i=0}^{5} t_i(u)\partial_u^i F(z,1)\,, \qquad (5)$$

where $r_k(u) := \{u^{<0}\}P(u)u^k$ and $t_i(u)$ are (Laurent) polynomials which can be made explicit.

One can use the kernel method (we refer to [4,9] for recent applications of this method) to solve this equation. We call $1-zP(u)$ the *kernel* of the equation. Solving $1-zP(u)=0$ with respect to u gives 4 roots $u_1(z)$, $u_2(z)$, $u_3(z)$ and $u_4(z)$ which are Puiseux series in $z^{1/4}$ and which tend to zero in 0. There are also 23 others roots which behave like $z^{-1/23}$ around 0, so we call $u_1,\dots,u_4$ the *small* roots of the kernel. Plugging the 4 small roots of the kernel in Equation 5 and considering the 6 other equations obtained by taking the first 5 derivatives of Equation 5 (and then setting $u=1$) gives a system of full rank with 10 equations with 10 unknown univariate generating functions, which are thus all algebraic, and then one has a formula for $F(z,u)$, involving the u_i, which implies its algebraicity. For the general case, simply replace 4 by b and 5 by d in Equation 5. Then, one can argue as in Theorem 7 above, with a new matricial equation $M.\overrightarrow{F} = \overrightarrow{v}$; looking at the valuation in z of each entries in M (some of them involves the small roots u_i's, but at most a product of b of them) gives $\det M \neq 0$ and thus a system of full rank, so $F(z,u)$ can be expressed as a rational function in z, u, and the small roots u_i's. As these roots are algebraic, $F(z,u)$ is algebraic. $\square$

3.6 Succession rules: "algebraic" exponents

Consider now the case where, for each i, the exponent $e(k,i)$ of the rule (2) is a constant (that is, $e(k,i) := e_{k-i}$ for a fixed sequence $(e_k)_{k\in\mathbb{Z}}$). "Algebraic" exponents means here that the generating function of the e_k's (which are nonnegative integers) is algebraic. How far can we relate the behaviour of the walk

$$[(0),(k) \rightsquigarrow (0)^{e_k}\dots(k-1)^{e_1}(k)^{e_0}(k+1)^{e_{-1}}\dots(k+a)^{e_{-a}}] \tag{6}$$

to the generating function of the exponents $E(u)=\sum_{i\geq -a} e_i u^i$? We give here a first element of answer:

Theorem 9 *Consider the generating tree*

$$[(0),(k) \rightsquigarrow (0)^{e_k}(1)^{e_{k-1}}\dots(k-1)^{e_1}(k)^{e_0}\dots(k+a)^{e_{-a}}]\,. \tag{7}$$

For $a=1$, one has

$$F(z,u)=\frac{F_0(z)}{1-u\,e_{-1}z\,F_0(z)} \qquad \text{with} \qquad F_0(z)=\frac{1}{e_{-1}z}\,E^{<-1>}(\frac{1}{z})$$

where $E^{<-1>}$ is the compositional inverse of $E(u)$ and where e_{-1} is the multiplicity of the $+1$ jump. More generally, for $a\geq 1$, the generating function $F(z,u)$ is expressed in terms of the a solutions $u_1(z),\dots,u_a(z)$ of $1-zE(u)=0$ which satisfy $u_k(z)\sim e^{2ik\pi/a}e_{-a}^{1/a}z^{1/a}$ for $z\sim 0$:

$$F(z,u)=F_0(z)\prod_{i=1}^{a}\frac{1}{1-uu_i(z)}=\sum_{k\geq 0}F_0(z)\left(\sum_{i_1+\dots+i_a=k}u_1^{i_1}\dots u_a^{i_a}\right)u^k\,.$$

One has

$$F_0(z)=\frac{(-1)^{a+1}}{ze_{-a}}\prod_{i=1}^{a}u_i(z) \qquad \text{and} \qquad F(z,1)=\frac{-1}{ze_{-a}}\prod_{i=1}^{a}\frac{1}{1-\frac{1}{u_i(z)}}\,.$$

Consequently, if the generating function of the exponents $E(u)$ is algebraic then the bivariate generating function $F(z,u)$ is algebraic.

Proof. For $a=1$, the first identity reflects the combinatorial decomposition (one to one correspondence, in fact) "a walk from 0 to $k+1$" is "a walk from 0 to k" then followed by a jump $+1$ then followed by "a walk from $k+1$ to $k+1$ never going below $k+1$". The generating function of these last walks is clearly $F_0(z)$, thus one has $F_{k+1}(z)=F_k(z)e_{-1}zF_0(z)=F_0(z)(ze_{-1}F_0(z))^{k+1}$.

For the walks corresponding to the rule (7), the set of jumps is given by $E(1/u)$; if one reverses the time direction, one gets a new walk where the set of available jumps is given by $E(u)$. Define $\widetilde{F}(z,u)$ as the corresponding generating function (one starts at altitude 0), one has:

$$\widetilde{f}_{n+1}(u) = \{u^{\geq 0}\}E(u)\widetilde{f}_n(u), \qquad \widetilde{f}_0(u) = 1$$

where $\{u^{\geq 0}\}$ stands for the sum of all monomials in u with a nonnegative degree. Multiplying by z^{n+1} and summing for $n \geq 0$ gives

$$\widetilde{F}(z,u) = \widetilde{f}_0(u) + zE(u)\widetilde{F}(z,u) - z\{u^{-1}\}\frac{e_{-1}}{u}\widetilde{F}(z,u)\,,$$

that one rewrites as the following functional equation

$$(1 - zE(u))\widetilde{F}(z,u) = 1 - z\frac{e_{-1}}{u}\widetilde{F}_0(z)\,.$$

Then solving the "kernel" $1 - zE(u) = 0$ with respect to u gives a series $u_1(z) = E^{<-1>}(1/z)$, which is algebraic as the compositional inverse of an invertible algebraic function is algebraic (simply plug the inverse in the polynomial equation $\Phi(E(u),u) = 0$ satisfied by $E(u)$ to check this fact). Note that E is invertible because $a \geq 1$ implies $E'(0) \neq 0$.

If one then evaluates the above functional equation at $u = u_1(z)$, one gets $0 = 1 - z\frac{e_{-1}}{u_1}\widetilde{F}_0(z)$ and thus $\widetilde{F}_0(z) = \frac{u_1}{e_{-1}z}$. As one has $\widetilde{F}_0(z) = F_0(z)$ (a walk from 0 to 0 from left to right is still a walk from 0 to 0 from right to left), one gets the result from the theorem. Note that if one sets $\widetilde{f}_0(u) = \frac{1}{1-u}$, $\widetilde{F}_0$ enumerates walks from anywhere to 0, so $\widetilde{F}_0(z) = \frac{u_1/(ze_{-1})}{1-u_1} = F(z,1)$, which is coherent with the theorem (case $a = 1$).

For $a \geq 1$, one sets $P(u) := \sum_{i=-a}^{-1} e_i u^i$; one has

$$(1 - zE(u))\widetilde{F}(z,u) = \widetilde{f}_0(u) - z\{u^{<0}\}P(u)\widetilde{F}(z,u)\,.$$

This is rewritten as

$$(1 - zE(u))\widetilde{F}(z,u) = \widetilde{f}_0(u) - z\sum_{k=0}^{a-1} r_k(u)\widetilde{F}_k(z)\,. \tag{8}$$

where $r_k(u) := \{u^{<0}\}P(u)u^k$ is a Laurent polynomial with monomials of degree going from -1 down to $k - a$.

$E(u)$ being algebraic, there exists a bivariate polynomial $P \in \mathbb{Q}[E,u]$ such that $P(E,u) = 0$. Now, as one has the kernel equation $1 - zE(u) = 0$, it means that the roots $u_i(z)$ of the kernel are algebraic and satisfy $P(\frac{1}{z}, u_i(z)) = 0$. The classical theory of Newton polygon then gives the Puiseux expansion of these roots. Among these roots, the kernel equation $1 - zE(u) = 0$ has a

roots $u_1(z), \ldots, u_a(z)$ which are Puiseux series in $z^{1/a}$ and which tend to 0 when z tends to 0. When $\widetilde{f}_0(u) = 1$, plugging these roots in the functional equation shows that they correspond to the a roots of the polynomial $u^a - zu^a \sum_{k=0}^{a-1} r_k(u) F_k(z)$, whose leading term is u^a and whose constant term is $-ze_{-a}\widetilde{F}_0(z)$. This gives $\widetilde{F}_0(z) = \frac{-\prod_{i=1}^{a} u_i}{-ze_{-a}}$. When $\widetilde{f}_0(u) = \frac{1}{1-u}$, this gives a system of a equations for a unknowns (the $\widetilde{F}_k$'s). Solving it for $\widetilde{F}_0$ gives $F(z,1)$. Solving the $\widetilde{F}_0$ for $\widetilde{f}_0(u) = u^k$ gives the $F_k(z)$. This concludes Theorem 9. □

Remark: as D-finite functions are not necessarily closed under compositional inverse, it is not true that if $E(u)$ is D-finite, then $F(z,1)$ or $F_0(z)$ (and *a fortiori* $F(z,u)$) are D-finite, even in the case $a = 1$.

For $a = 1$, the Riordan arrays approach that we presented in Subsection 2.3 also gives the algebraicity of $F(z,u)$. In fact, a theorem from [22] says:

Theorem 10 *If $(a_j)_{j\in\mathbb{N}}$ and $(z_j)_{j\in\mathbb{N}}$ are two nonnegative integer sequences, with $a_0 \neq 0$, then the matrix associated to the generating tree*

$$\begin{cases} (r) \\ (k) \rightsquigarrow (r)^{z_{k-r}} (r+1)^{a_{k-r}} (r+2)^{a_{k-r-1}} \ldots (k+1)^{a_0} \end{cases} \tag{9}$$

is a proper Riordan Array D defined by the triple (d_0, A, Z), such that

$$d_0 = 1, \quad A = (a_0, a_1, a_2, \ldots), \quad Z = (z_0, z_1, z_2, \ldots).$$

Accordingly, this gives

$$F(z,u) = \frac{d(z)}{1 - uzh(z)} \text{ where } h(z) = A(zh(z)) \text{ and } d(z) = 1/(1 - zZ(zh(z)))\,.$$

For $a > 1$, the matrix associated (see Section 2) to the rule (6) is called a *horizontally stretched Riordan array.* The algebraicity of the corresponding generating function $F(z,u)$ then depends on the algebraicity of $A(z) = \sum_{k\geq 0} a_k z^k$ and $F_0(z), \ldots, F_{a-1}(z)$ (the generating functions of the first a columns of the matrix). However, while the theory of Riordan arrays has been intensively studied, the theory of stretched Riordan arrays, from a generating function point of view, is still in progress.

We end with a last application of the kernel method.

Theorem 11 *Consider the succession rule (6) when the e_i's are ultimately constants (say, equal to a constant C after rank b):*

$$[(0), (k) \rightsquigarrow (0)^C \ldots (k-b-1)^C (k-b)^{e_b} \ldots (k)^{e_0} \ldots (k+a)^{e_{-a}}]\,.$$

Then $F(z,u)$ is algebraic and satisfies

$$F(z,u) = \frac{\prod_{i=0}^{b} u - u_i(z)}{K(z,u)}\,,$$

where the u_i's and K are defined as below.

Proof. One has the recurrence $f_{n+1}(u) = C\frac{f_n(u)-f_n(1)}{u-1} + P(u)f_n(u)$ this leads to the functional equation

$$\begin{aligned}\left(1 - zP(u) - z\frac{C}{u-1}\right) F(z,u) \quad &= \\ 1 - \frac{zC}{u-1}F(z,1) - z\sum_{k=0}^{b-1}\{u^{<0}\}P(u)u^k F_k(z)\,, \quad &\end{aligned} \tag{10}$$

where $P(u) = \sum_{i=1}^{b}(e_i - C)\frac{1}{u^i} + \sum_{i=0}^{a} e_{-i}u^i$. Define the kernel K as $K(u,z) = u^b(1-u)(1 - zP(u) - \frac{zC}{u-1})$. It has b roots $u_1(z),\dots,u_b(z)$ which are Puiseux series in $z^{1/b}$ and which tend to 0 in 0 and one root $u_0(z)$ which tends to 1 in 0. These are exactly the $b+1$ roots of the right hand part of (10) (once multiplied by $(1-u)u^b$). So $F(z,u) = \frac{\prod_{i=0}^{b} u-u_i(z)}{K(z,u)}$, where the u_i's are the $b+1$ small roots of the kernel. $\square$

3.7 Asymptotics

Given a particular rule for Theorems 7, 8, 9, 10 or 11, it is possible to find an asymptotic expansion for the number of walks. It is not really possible to merge all these results in a single one, as the rules are too unconstrained. However, for the algebraic case, a kind of universality holds for the behaviour of the roots of the kernel. This leads to following theorem, which has to be adapted case by case for rules of Theorems 8 and 9 (and is easily applied to rules of Theorem 11).

Theorem 12 *The number of walks of length n for the "factorial" rule*

$$[(0),(k) \rightsquigarrow (0)(1)\dots(k-b-1)(k-b)^{e_b}\dots(k)^{e_0}\dots(k+a)^{e_{-a}}]$$

(where $e(k,i) = 1$ for $0 \le i < k-b$ and $e(k,i) = e_{k-i}$, some fixed constants, for $i \ge k-b$) has the following asymptotics $A\frac{\rho^{-n}}{\sqrt{2\pi n^3}}$, where A and ρ are algebraic constants depending on the finite multiset of jumps $\mathcal{P} = \{-b,\dots,+a\}$.

Proof. See [2] for a proof and applications to the limit laws of final altitude and number of factors. The approach is similar to the one used for walks with a finite number of jumps but there are some complications due to the fact that the kernel is now of the kind $1 - z\phi(u)$ where $\phi(u)$ is not unimodal. One can however establish that the real positive root u_0 now dominates and has a square-root behaviour. $\square$

This result is the first step towards limit laws of several parameters (like final altitude, local time, ...). It would be interesting (but much more difficult) to get the asymptotics of parameter like height and area. Note that for these parameters, it is possible to get closed-form formulae (particularly in the case $a = 1$ for the area and for any value of a for the height, via the kernel method, see [1])... but this is another story!

3.8 Algebraic equations

In Theorems 3 and 4, we gave a direct way to obtain an algebraic equation satisfied by $F(z, 1)$ when the generating function of the exponents is rational. For the other theorems, as the algebraic generating function $F(z, u)$ is expressed in terms of the roots of the kernel, it is possible to get an algebraic equation for $F(z, u)$ via resultant or Gröbner bases computations. Note that a more efficient way, the so-called Platypus algorithm, is presented in [4]. It also relies on an exploitation of the roots of the kernel.

3.9 Variations...

As a first variation, it is possible to play with the root r of the tree (the starting point of the paths). We gave above results mostly for $r = 0$ or $r = 1$, but it is also possible to follow our proofs for other values of r.

As a next variation, it is also possible to remove the non-negativity constraint. In this case, the walks are on $\mathbb{Z}$ and thus one gets directly $F(z, u) = \frac{1}{1-zE(u^{-1})}$. If one then considers walks ending at a given altitude k, it is possible to get a closed-form formula for their generating function $F_k(z)$ (which is algebraic), via residue computation and a conjugacy principle (simply follow the same proofs as in [2,4] and get Spitzer-like formulae and closed-form formulae still involving the roots of the kernel).

As a third variation, in Rule (2), it also possible to consider exponents $e(k, i)$ from $\mathbb{Z}^2$ to $\mathbb{Z}$, even if the combinatorial meaning of a "negative multiplicity" is not clear... It is also possible to consider the case $e(k, i) = e_i$ (instead of $e(k, i) = e_{k-i}$ as we did in this article). The sequences increase very quickly, it is then then natural to look for exponential generating functions. Some nice formulae were given in [1,3] and combinatorial proofs were given in [11].

As a last variation, it is also possible (see [1]) to reconsider all the above results for walks of higher Markovian order, that is for walks for which $f_{n+1}(u)$ depends not only on $f_n(u)$ but also on $f_{n-1}(u)$, and on finitely many other f_n's. Here again, our approaches are still working. For example, with a Markovian random walk of order 2 (positions one step before are involved with multiplicities encoded by $E(u)$, and positions two steps before are involved with multiplicities encoded by $E_2(u)$), formulae roughly involves something like $\frac{A}{1-zE(u)-z^2E_2(u)}$ instead of $\frac{A}{1-zE(u)}$. There are already combinatorial interests for such walks, see [18].

4 Examples

We now give a series of examples from combinatorics or computer science in which succession rules studied in Section 3 appear.

EXAMPLE 1. *Fully directed compact animals.*

They are also called *Diagonally directed convex polyominoes* (see [16] and Fig. 6). They are known to be counted according to their number of diagonals by $\frac{1}{2n+1}\binom{3n}{n}$ which corresponds to the the generating tree $[(1),(k) \rightsquigarrow (1)^{k+1}(2)^k \dots (k-1)^3(k)^2(k+1)]$. $\square$

EXAMPLE 2. *A new generating tree for Catalan numbers.*

From [24] (see the exercise on Catalan numbers pp.221-247), the generating tree

$$[(1),(k) \rightsquigarrow (1)^{2^{k-2}}(2)^{2^{k-3}} \dots (k-2)^2(k-1)(k+1)]$$

generates the partition $\{B_1,\dots,B_p\}$ of $[n]$ such that the numbers $1,2,\dots,n$ are arranged in order around a circle, then the convex hulls of the blocks $B_1,\dots,B_p$ are pairwise disjoint. Indeed, let k be the number of isolated points around 1. The 2^{k-1} successors of this configuration are obtained by taking all the subset of $\{\alpha_1 = 1, \alpha_2, \dots, \alpha_k, n+1\}$ containing $n+1$. $\square$

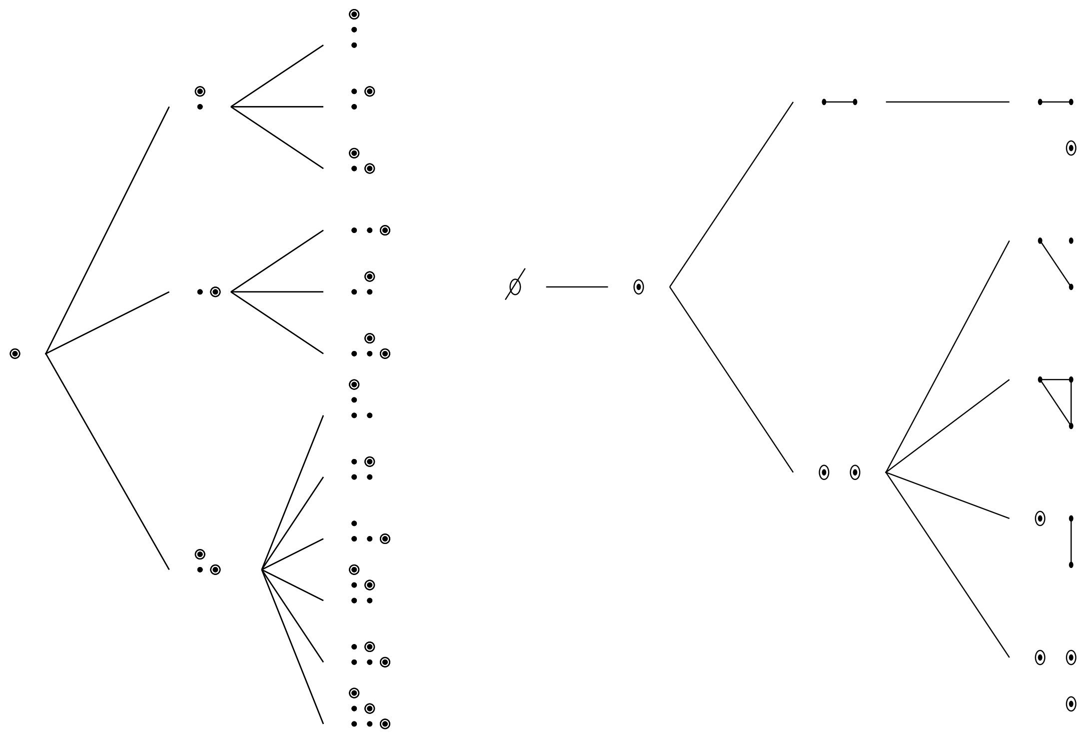

Fig. 6. Generating trees for (fully directed compact) animals and Catalan blocks.

EXAMPLE 3. *Two families of rules leading to an algebraic generating function.*

For the rule $[(0),(k) \rightsquigarrow (0)^{e_k}(1)^{e_{k-1}}\dots(k-1)^{e_1}(k)^{e_0}(k+1)]$, where e_k for $k \geq 0$ is the number of t-ary trees with k nodes, $F(z,u)$ satisfies a algebraic equation of degree t. E.g., for $t=3$, one has:
$1-(3+(4-3u)z)F(z,u)-(-3+(6u-7)z+(-3u^2+8u-3)z^2)F(z,u)^2-(1+(3-3u)z+(3u^2-7u+3)z^2+(-u^3+4u^2-3u+1)z^3)F(z,u)^3=0.$

For the rule $[(0),(k) \rightsquigarrow (0)^{c+k}(1)^{c+k-1}\dots(k-2)^{c+2}(k-1)^{c+1}(k)^{c}(k+1)]$, $F(z,u)$ satisfies an algebraic equation of degree 3:
$((1-2u)z^2+(c-(c+1)+2u^2))F^3+((u-2)z+(-c-2+4u-2u^2)z^2)F^2+(1+(2-2u)z)F=1.$

Similar examples for $a>1$ lead to expressions which are perhaps a bit large to be written here *in extenso*. However, the reader interested by such examples can have a look at `http://algo.inria.fr/banderier/Papers/dm03.mws`. This is a Maple worksheet where we get the equations for $F(z,u)$, plot the roots of the kernel, give the asymptotics for different kind of walks. □

EXAMPLE 4. *Tennis ball problem.*

Let $s \geq 2$ be an integer and consider the following problem known as *the s-tennis ball problem*. At the first turn one is given balls numbered 1 to s. One throws one of them out of the window onto the lawn. At the second turn balls numbered $s+1$ through $2s$ are brought in and now one throws out on the lawn any of the $2s-1$ remained. Then balls $2s+1$ through $3s$ are brought in and one throws out one of the $3s-2$ available balls. The game continues for n turns. At this point, one picks up the n balls in the lawn and consider the ordered sequence $B=(b_1,b_2,\dots,b_n)$ with $b_1<b_2<\dots<b_n$. This sequence is called a *tennis ball s-sequence* and the first question is: how many tennis ball s-sequences of length n exist? The second question is: what is the sum of all the balls in all the possible s-sequences of length n ? Obviously, if we answer to both these questions, we also know the average sum of the balls in an s-sequence of length n. The general case $s \geq 1$ has been studied in [21] from a generating function viewpoint. In fact, the authors consider an infinite tree with root 0 and with s children. Each $(n+1)$-length path in this tree corresponds to an s-sequence of length n. This infinite tree is isomorphic to the generating tree with specification $[(1),(k) \rightsquigarrow (1)\dots(k+s-2)(k+s-1)]$.

By using this result the authors find that the number of tennis ball s-sequences of length n are counted by T_{n+1}, where $T_n=\frac{1}{1+(s-1)n}\binom{sn}{n}$ (the number of s-ary trees with n-nodes) and the cumulative sum of all the balls thrown onto the lawn in n turn is

$$\Sigma_n = \frac{1}{2}(sn^2 + (3s-1)n + 2s)T_{n+1} - \frac{1}{2}\sum_{k=0}^{n+1}\binom{sk}{k}\binom{s(n+1-k)}{n+1-k}.$$

□

Example 5. A new succession rule for $(4,2)$-tennis ball problem.

The problem of balls on the lawn admits many other variants. For example, one could be supplied with s balls at each turn but now throw out t balls at a time with $t < s$. The general (s,t) case is an open problem while the $(4,2)$ case has been treated in [21], where the authors study the problem by introducing a bilabelled generating tree technique. Anyway, recently Merlini and Sprugnoli found that the problem can be expressed by the rule (6) with $e_i = i+3$ and $a = 2$, namely:

$$[(0),(k) \rightsquigarrow (0)^{k+3}(1)^{k+2}(2)^{k+1}\dots(k+2)] \tag{11}$$

In fact, if we don't care of the order of the balls thrown away, so that the configuration $(1,4)$, $(5,8)$, $(2,10)$ is considered to be the same as $(1,2)$, $(4,5)$, $(8,10)$, it can be proved that the number of $(4,2)$-sequences of length $2n$ in which the last-but-one element is $2n+k-1$ corresponds to the number of nodes with label k at level n in the generating tree of Figure 7 (for example, the possible sequences of length 2 are $(1,2)$, $(1,3)$, $(1,4)$, $(2,3)$, $(2,4)$ and $(3,4)$). □

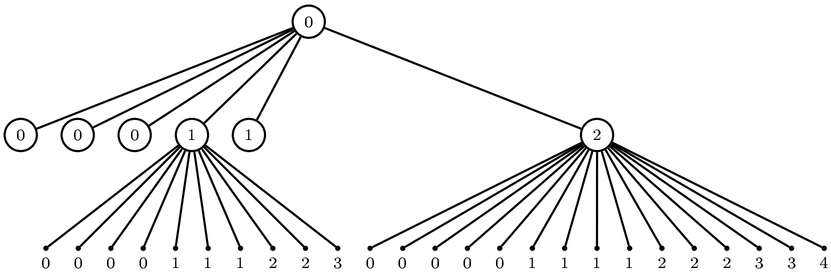

Fig. 7. The partial generating tree for the specification (11).

Example 6. *Printers.*

In [20] the authors present a combinatorial model for studying the characteristics of job scheduling in a slow device, for example a printer in a local network. The policy usually adopted by spooling systems is called *First Come First Served* (FCFS) and can be realised by queueing the processes according to their arrival time and by using a FIFO algorithm. A job (printing a file) consists in a finite number of *actions* (printing-out a single page). Each action takes constant time to be performed (a *time slot*). If we fix n time slots, and suppose that at the end of the period the queue becomes empty, while it was never empty before, the successive states of the jobs queue can be described by a combinatorial structure called *labelled 1-histograms*. A 1-*histogram* of length n is a histogram whose last column only contains 1 cell and, whenever a column is composed by k cells, then the next column contains at least $k-1$ cells. It is at all obvious that a 1-histogram corresponds to a path in the generating tree produced by the specification $[(1),(k) \rightsquigarrow (1)\dots(k+1)]$. A *labelled* 1-*histograms* of length n is a 1-histogram in which we label each cell according to some rules (see [20] for the details). Figure 8 illustrates the possible schedules for two particular 1-histograms of length 3: the first one, for example, corresponds to i) a first job which consists in printing two pages and a second job, which starts at time slot 2, and corresponds to printing a page at time slot 3, and ii) three different jobs which consist in printing a single page, the first at time slot 1, the second at time slot 2 and the third at time slot 3, after queueing at time slot 2. It can be proved that the number of schedules of length n with k jobs request at the first time slot corresponds to the number of nodes at level n having label $k+1$ in the generating tree with specification:

$$[(1),(k) \rightsquigarrow (1)^2 \dots (k)^2 (k+1)]\,.$$

This gives that the number S_n of possible schedules corresponds to the n^{th} small Schröder number, that is, the generating function for S_n is $(1-3z-\sqrt{1-6z+z^2})/(4z)$. $\square$

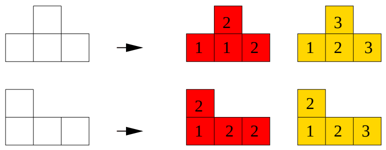

Fig. 8. The schedules corresponding to two particular 1-histograms.

Acknowledgements. Cyril Banderier's work was partially supported by the Future and Emerging Technologies programme of the EU under contract number IST-1999-14186 (ALCOM-FT), by the INRIA postdoctoral programme and by the Max-Planck Institut. He is now supported by the CNRS. During their invitations in Florence, the three French authors also benefited of pasta. Finally, the four authors are grateful to anonymous referees (from the FPSAC'02 conference) for their constructive comments on former versions [5,15] of this article.

Rule	EIS description	Generating Function $F(z,u)$
$(0),(k) \leadsto (0)^k(k+1)$	F_0, $F(z,1)$: powers of 2	$\dfrac{1-2z-z^2}{1-(u+2)z-2uz^2}$
$(0),(k) \leadsto (0)^{2k}(k+1)$	$F(z,1)$:A001333 continued fraction convergents to $\sqrt{2}$ F_0: A052542 (ECS)	$\dfrac{1-2z+z^2}{1-(u+2)z+(2u-1)z^2+uz^3}$
$(0),(k) \leadsto (0)^{3k}(k+1)$	$F(z,1)$: A026150 (ECS)	$\dfrac{1-2z+z^2}{1-(u+2)z+(2u-2)z^2+2uz^3}$
$(0),(k) \leadsto (0)^{4k}(k+1)$	$F(z,1)$: A046717 half of 3^n	$\dfrac{1-2z+z^2}{1-(u+2)z+(2u-3)z^2+3uz^3}$
$(0),(k) \leadsto (0)^k(k+1)(k+2)$	$F(z,1)$: A001075 and F_0: A005320 Pell's equation	$\dfrac{1-4z+4z^2}{1-(4+u+u^2)z+(4u^2+u-1)z^2-\dots}$
$(1),(k) \leadsto (0)(1)^2(k)(k+2)^2(k+3)^5$	6^n and A003464 $(6^n-1)/5$	$\dfrac{(4u-1)z-u}{(1-6z)((2u^2+1)z-1)}$
$(0),(k) \leadsto (0)^{k^2}(2)^{3k-1}(3)(k)(k+1)^2(k+3)^5$		see Theorem 7

Table 1
Some succession rules leading to rational generating functions. The generating functions $F(z,1)$ and $F_0(z)$ are defined as in Equation 1.

Rule	EIS description	Generating Function $F(z,u)$
$(1),(k) \rightsquigarrow (1)\dots(k+s-2)(k+s-1)$	$F(z,1)$: s-ary trees	See also Ex. 4
$(1),(k) \rightsquigarrow (1)^2\dots(k)^2(k+1)$	$F(z,1)$: A001003 Schröder's second problem	$\frac{u}{2}\frac{1-(2u+1)z-\sqrt{1-6z-z^2}}{(1-u)z+(u^2+u)z^2}$ (see also Ex. 6)
$(0),(k) \rightsquigarrow (0)^{k^2}(2)^{3k-1}(3)(k-1)(k)(k+1)^2(k+3)^5$		see Theorem 8
$(0),(k) \rightsquigarrow (0)^k(1)^{k-1}\dots(k-1)^1(k)^0(k+1)$	A036765 $F(z,1)$: rooted trees with a degree constraint	equation of degree 3
$(0),(k) \rightsquigarrow (0)^{k+2}(1)^{k+1}\dots(k-1)^3(k)^2(k+1)$	F_0: A006013 A046648 noncrossing trees on a circle $F(z,1)$: A001764 ternary trees	equation of degree 3 (see Ex. 1 for a variant)
$(0),(k) \rightsquigarrow (0)^{k+3}\dots(k-1)^4(k)^3(k+1)^2(k+2)$	$F(z,1)$: A066357 planar trees with root parity constraint	equation of degree 4 (see also Ex. 5)
$(0),(k) \rightsquigarrow (0)^{C_k}\dots(k-1)^{C_1}(k)^{C_0}(k+1)$ (where C_k is the k-th Catalan number)	F_0: A006318 large Schröder numbers	$\frac{1}{2}\frac{3-(4u+1)z-\sqrt{1-6z-z^2}}{1-3uz+(2u^2+u)z^2}$
$(0),(k) \rightsquigarrow (0)^{C_k}\dots(k-1)^{C_1}(k+1)$	F_0: A052705 (ECS)	$\frac{1}{2}\frac{3-(4u+2)z-\sqrt{1-4z-4z^2}}{1-(3u+2)z+(2u^2-2u+1)z^2}$
$(0),(k) \rightsquigarrow (0)^{T_k}\dots(k-1)^{T_1}(k)^{T_0}(k+1)\}$ (where T_k is the k-th tri-Catalan number)	F_0: A054727 noncrossing forests of rooted trees	equation of degree 3 (see Ex. 3)

Table 2
Some succession rules leading to algebraic generating functions.

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