WHY DELANNOY NUMBERS?

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Abstract. This article is not a research paper, but a little note on the history of combinatorics: We present here a tentative short biography of Henri Delannoy, and a survey of his most notable works. This answers to the question raised in the title, as these works are related to lattice paths enumeration, to the so-called Delannoy numbers, and were the first general way to solve Ballot-like problems.

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1. Classical lattice paths

Before to tackle the question of Delannoy numbers and Delannoy lattice paths, note that the classical number sequences or lattice paths have the name of a mathematician: the Italian Leonardo Fibonacci ($\sim$1170–$\sim$1250), the French Blaise Pascal (1623–1662), the Swiss Jacob Bernoulli (1654–1705), the Scottish James Stirling (1692–1770), the Swiss Leonhard Euler (1707–1783), the Belgian Eugène Catalan (1814–1894), the German Ernst Schröder (1841–1902), the German Walther von Dyck (1856–1934), the Polish Jan Łukasiewicz (1878–1956), the American Eric Temple Bell (1883–1960), the American Theodore Motzkin (1908-1970), the Indian Tadepalli Venkata Narayana (1930–1987), . . . It is quite amusing that some of them are nowadays more famous in combinatorics for problems which can be explained in terms of lattice paths than in their original field (algebra or logic for Dyck, Schröder, and Łukasiewicz[1]).

Fibonacci numbers appear in his 1202 *Liber abaci* (also spelled *abbaci*) [80]. "Catalan numbers" can be found in various works, including [17, 79]. Catalan called these numbers "Segner numbers"; and the actual terminology is due to Netto who wrote the first classical introduction to combinatorics [68]. The name "Schröder numbers" honors the seminal paper [76] and can be found in Comtet's "Analyse combinatoire" [21] and also in one of his articles published in 1970. The name "Motzkin numbers" can be found in [37] and is related to Motzkin's article [66]. The name "Narayana numbers" was given by Kreweras by reference to the article [67] (these numbers were also independently studied by John P. Runyon, a colleague of Riordan. These are called Runyon numbers in Riordan's book [73], p.17). The name "Dyck paths" comes from the more usual "Dyck words/Dyck Language" which have been widely used for more than fifty years. We strongly recommend the lecture of R. Stanley, which gives some comments about the surprisingly old origin of these names and problems (cf pp. 212–213 of [81]).

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[1]See the MacTutor History of Mathematics `http://www-groups.dcs.st-and.ac.uk/~history/`

2. Delannoy numbers

Delannoy is another "famous" name which is associated to an integer sequence related to lattice paths enumeration. Delannoy's numbers indeed correspond to the sequence $(D_{n,k})_{n,k\in\mathbb{N}}$, the number of walks from $(0,0)$ to (n,k), with jumps $(0,1)$, $(1,1)$, or $(1,0)$.

1	19	181	1159	5641	22363	75517	224143	598417	**1462563**
1	17	145	833	3649	13073	40081	108545	**265729**	598417
1	15	113	575	2241	7183	19825	**48639**	108545	224143
1	13	85	377	1289	3653	**8989**	19825	40081	75517
1	11	61	231	681	**1683**	3653	7183	13073	22363
1	9	41	129	**321**	681	1289	2241	3649	5641
1	7	25	**63**	129	231	377	575	833	1159
1	5	**13**	25	41	61	85	113	145	181
1	**3**	5	7	9	11	13	15	17	19
1	1	1	1	1	1	1	1	1	1

In this array, the lower left entry is $D_{0,0} = 1$ and the upper right entry is $D_{10,10} = 8097453$. Entry with coordinates (n,k) gives the number of Delannoy walks from $(0,0)$ to (n,k). The three steps $(0,1)$, $(1,1)$, and $(1,0)$ being respectively encoded by x, y and xy, the generating function of Delannoy walks is

$$F(x,y,t) = \sum_{n\geq 0}(x+y+xy)^n t^n = \frac{1}{1-t(x+y+xy)}\,,$$

where t encodes the length (number of jumps) of the walk.

The *central* Delannoy numbers $D_{n,n}$ (`EIS 1850`[2]) are in bold in the above array. They have appeared for several problems: properties of lattice and posets, number of domino tilings of the Aztec diamond of order n augmented by an additional row of length $2n$ in the middle [75], alignments between DNA sequences [86]...The generating function of the central Delannoy numbers is

$$\begin{aligned} D(z) &:= \sum_{n\geq 0} D_{n,n} z^n = [x^0]\,\frac{1}{1-(zx+z/x+z)} = \frac{1}{\sqrt{1-6z+z^2}} \\ &= 1+3\,z+13\,z^2+63\,z^3+321\,z^4+1683\,z^5+8989\,z^6+48639\,z^7+O(z^8)\,. \end{aligned}$$

The notation $[x^n]F(x)$ stands for the coefficient of x^n in the Taylor expansion of $F(x)$ at $x=0$. The square-root expression is obtained by a resultant or a residue computation (this is classical for the diagonal of rational generating functions). This closed form for $D(z)$ gives, by singularity analysis (see the nice book [43]):

$$\begin{aligned} D_{n,n} &= \frac{(3+2\sqrt{2})^n}{\sqrt{\pi}\sqrt{3\sqrt{2}-4}}\left(\frac{n^{-1/2}}{2} - \frac{23\,n^{-3/2}}{32(8+3\sqrt{2})} + \frac{2401\,n^{-5/2}}{2048(113+72\sqrt{2})} + O(n^{-7/2})\right) \\ &\approx 5.82842709^n\left(.57268163\,n^{-1/2} - .06724283\,n^{-3/2} + .00625063\,n^{-5/2} + \dots\right). \end{aligned}$$

One has also $D_{n,k} = \sum_{i=0}^{n}\binom{n}{i}\binom{k}{i}2^i$. Quite often, people note that there is a link between Legendre polynomials and Delannoy numbers [46, 56, 64], and indeed

[2]This number refers to the wonderful On-Line Encyclopedia of Integer Sequences, see `http://www.research.att.com/~njas/sequences/`

$D_{n,n} = P_n(3)$, but this is not a very relevant link as there is no "natural" combinatorial correspondence between Legendre polynomials and these lattice paths. Comtet [21] showed that the coefficients of any algebraic generating function satisfy a linear recurrence (which allows to compute them in linear time). For $d_n := D_{n,n}$, it leads to $(n+2)d_{n+2} - (6n+9)d_{n+1} + (n+1)d_n = 0$.

For this kind of lattice paths with jumps $-1, 0, +1$, one has links with continuous fraction [42], with determinants [45], with context free grammars [55]... Numerous generalizations have been investigated: walks in the quarter plane [39], multi-dimensional lattice paths [6, 52, 82, 47, 48].

It is classical in probability theory (see [11] for some discussions with a combinatorial flavor) and more precisely in the theory of Brownian motion to consider the following constraints for lattice paths:

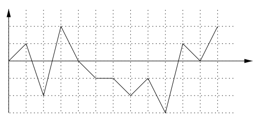
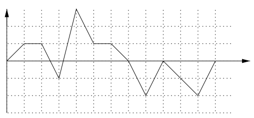
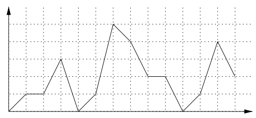
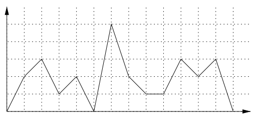
walks	ending anywhere	ending in 0
unconstrained (on $\mathbb{Z}$)	 walk ($\mathcal{W}$)	 bridge ($\mathcal{B}$)
constrained (on $\mathbb{N}$)	 meander ($\mathcal{M}$)	 excursion ($\mathcal{E}$)

FIGURE 1. The four types of paths: walks, bridges, meanders, and excursions.

For these four kinds of walks and for any finite set of jumps, there exists a nice formula for the corresponding generating function, which appears to be algebraic and from which one can derive the asymptotics and limit laws for several parameters of the lattice paths (see [9]).

Delannoy numbers $D_{n,n}$ correspond to bridges with a set of jumps $\{+1, -1, 0\}$ (where the 0 jump is in fact of length 2). Consider now the language $\mathcal{D}$ of central Delannoy paths, encoded via the letters a, b, c (for the jumps $+1, -1, +0$ resp.). Excursions with these jumps are called Schröder paths. We note $\mathcal{S}$ the language of Schröder paths (excursions) and $\bar{\mathcal{S}}$ the set of their mirror with respect to the x-axis. Then, the natural combinatorial decomposition $\mathcal{D} = (c^* a\mathcal{S}b + c^* b\bar{\mathcal{S}}a)^* c^*$ (which means that one sees a Delannoy path [bridge] as a sequence of Schröder paths [excursions] above or below the x-axis) leads to

$$D(z^2) = \frac{1}{1 - 2z^2 S(z)\frac{1}{1-z^2}} \frac{1}{1-z^2} \text{ where } S(z) = \frac{1 - z^2 - \sqrt{1 - 6z^2 + z^4}}{2z^2}$$

is the generating function of Schröder paths. This link between excursions and bridges is always easy to express when the set of jumps is symmetric or with jumps of amplitude at most 1, but there is also a relation between excursions and bridges in a more general case (see [9] for combinatorial and analytical proofs).

FIGURE 2. Henri Auguste Delannoy (1833-1915). Portrait provided by the Société des Sciences Naturelles et Archéologiques de la Creuse, where it is exhibited.

Despite all these appearances of Delannoy numbers (see [84] for a list of 29 objects counted by Delannoy numbers!), the classical books in combinatorics or computer science which are usually accurate for "redde Caesari quae sunt Caesaris" (e.g., Comtet, Stanley, Knuth) are mute about this mysterious Delannoy.

3. Henri Auguste Delannoy (1833-1915)

Some people suggested that "Delannoy" was related either to the French mathematician Charles Delaunay (like in Delaunay triangulations) or to the Russian mathematician Boris Nikolaevich Delone, but this is not the case, as we shall see.

It is true that "Delannoy" sounds like (and actually is) a French family name `[d∧lanoa]` in approximative phonetic alphabet (∧ like in "duck" and `a` in "have"). There are in fact thousands of Delannoy, mostly in the North of France and in Belgium. This toponym means "de Lannoy", that is to say who originates from the town of Lannoy; "lannoy" meaning a place with a lot of alders; yet how to find "our" Delannoy among all these homonyms? The terminology "Delannoy numbers" became widely used as it can be found in Comtet's book in the footnote from exercise 20, p.93 [21]: "these numbers are often called Delannoy numbers" without any reference. In the English edition "Advanced Combinatorics" [22], the footnote becomes inserted in the text (p.81) but there is still no reference.

In fact, it appears to be a good idea to look in Lucas' books (see [24] for some biographical informations on Édouard Lucas [1842-1891]): Indeed, in the second edition of the first volume of the *Récréations mathématiques* [58], Lucas wrote in the preface "*J'adresse mes plus vifs remerciements à mon ami sincère et dévoué, Henry*[3] *Delannoy,...*" and at page 13 of this introduction we are told that Henri Delannoy was intendant. In the second volume [63], the fourth recreation was dedicated to *Monsieur Henri Delannoy, ancien élève de l'École Polytechnique, sous-intendant militaire de Première classe*. After the death of his friend Lucas in October 1891, Henri Delannoy contributed with Lemoine and Laisant[4] to the publication of the third and fourth volumes[5] [60, 61] as well as to the book *L'arithmétique amusante* [62].

Like most of the French military intendants, Delannoy was a student from the École Polytechnique (which was the place where military officers received a scientific education). From a database[6] of the former students, one knows that Henri (Auguste) Delannoy is born in 1833 in Bourbonne-les-Bains (Haute-Marne, France). His father was Omère Benjamin Joseph Delannoy (countable officer) and his mother was Françoise Delage; they were living in the city of Bourges. In 1853, he passed the École Polytechnique entrance exam (with rank 62); then he graduated in 1854 with rank 91/106 and finished with rank 67/94 in 1855. It is quite funny that this database also contains, like for any other polytechnician, a physical description of Henri Delannoy: dark brown hair, average brow, average nose, blue eyes, small mouth, round chin, round face, height: 1,68m!

In the archive center of the French Army (in the Château de Vincennes), one can find his record under the number 61241. From this and [1, 7], we know that Delannoy was first in the Artillery corps as sous-lieutenant (with rank 12/37 from the application school), lieutenant (1857), took part in the Italy campaign (27 May-18 August 1859) and in the Solférino battle (24 June 1859). When he came back, he married his dulcinea Olympe-Marguerite Guillon on the 10 November 1859. They had 2 daughters and one boy. Delannoy was promoted captain in 1863. He then became a supplier-administrator: Intendant-Adjoint in 1865, sous-intendant of third class in 1867, of second class in 1872, of first class in 1882 (he was now a widower). He spent three years in Africa (6 Oct. 1866 - 25 Oct. 1869). He was the governor of the military Hospital of Sidi-bel-Abes, Algeria, during the terrible typhus epidemic (he belonged the Supply Corps and they were in charge the sanitary affairs). He translated for himself and perhaps also for his hierarchy several German books/notes about the Supply Corps. He took part in the 1870 war between France and Prussia. It is mentioned without explanation that he was in *Deutschland* on July 26, 1870 (that is, 4 days after the declaration of war...) and on March 7, 1871 (that is, 3 days before the treaty of London...). He was decorated with the *médaille d'Italie*, the *décoration sarde de la Valeur militaire*, the *Croix de la Légion d'Honneur* on July 18, 1868, and the *Rosette d'Officier de la Légion d'Honneur* in December 20, 1886. He could have reached the highest military ranks, but he wanted peace and decided to retire (January 9, 1889) in Guéret (the main city of the French department "la Creuse"), beginning a second life dedicated to science and more particularly to mathematics.

[3]There is no mistake, he was born Henry and asked to change it into Henri. We use in this article the first name Henri, as it was Delannoy's choice and as this was officially approved.

[4]The rôle played by each one is explained in [4].

[5]Available at the web site of the French National Library `http://gallica.bnf.fr/`

[6]Available at `http://bibli.polytechnique.fr:4505/ALEPH0/`

Looking for `Delannoy*` in the Zentralblatt volumes of 1860–1920[7], gives the following nine articles [26, 27, 28, 29, 31, 34, 33, 35, 36]. The references, problems, methods, and solutions used in these articles are similar to the ones often mentioned in Lucas' books. Most of Delannoy's articles are signed by Monsieur (H.) Delannoy, military intendant in the city of Orléans (and later, retired military intendant in the city of Guéret). Delannoy was a quite active member of the French Mathematical Society (SMF) in which he was admitted in 1882, introduced by Lucas and Laisant. He disappeared from the SMF's list in 1905, while he still contributed to l'Intermédiaire des Mathématiciens until 1910. This amateur mathematician then sank into oblivion and we found no obituary in the SMF bulletins at the occasion of his death on February 5, 1915.

In his death certificate, he was referred as president of the *Société des Sciences Naturelles et Archéologiques de la Creuse*. This Society is in fact still very dynamic[8]. It eventually appears that this Society, over which Delannoy presided from 1896 to 1915, has some archives, a part of which was given by Delannoy's family. They include some biographies written when Delannoy was still alive [10, 1], a list of his publications, and also an obituary and a short biography by members of the Society [7, 15]. We shall come back on Delannoy's works in this Society in Section 5 and we now consider Delannoy's contribution to mathematics.

4. Delannoy's mathematical work

Delannoy began his mathematical life reading the mathematical recreations that Lucas began to publish in 1879 in *La Revue Scientifique*. He was in contact with him in 1880 and began immediately to work with him, answering to letters of mathematicians transmitted by Lucas.

The first mention, in a mathematical work, to Delannoy is in an article by Lucas "Figurative arithmetics and permutations (1883)" [57], which deals with enumeration of configurations of 8 queens-like problems (the simplest one being: how to place n tokens on an $n \times n$ array, with no row or column with 2 tokens). Delannoy is there credited for having computed several sequences.

Some years later, in 1886, Delannoy made his first mathematical public appearance in the annual meeting of the "Association Française pour l'Avancements des Sciences". We now give the list of Delannoy's articles.

4.1. **Using a chessboard to solve arithmetical problems(1886)** [25]. In this article, Delannoy comes back on Lucas' article mentioned above and explains how he can use a "chessboard" (in modern words: an array) to get the formula $\frac{n-k+1}{n+1}\binom{n}{n+k}$ for the number of Dyck paths of length n ending at altitude k by using something which is not far from what one calls now the Desiré André reflection principle, which was in fact published one year later [3]. Note that Feller says (without references, see pages 72 and 369 of [41], 340 of [40]) that Lord Kelvin "method of images" for solving some partial differential equations is a kind analytic equivalent of the reflection principle in disguise. However, if one has a look on William Thomson's letter to Liouville [85], the link is rather mild and the cleaver combinatorial ideas of André & Delannoy cannot be attributed to Lord Kelvin.

[7]The "Jahrbuch über die Fortschritte der Mathematik" (annual review on the progresses of mathematics) is available at `http://www.emis.de/MATH/JFM/`

[8]`htpp://perso.wanadoo.fr/jp-l/SSC23/`

Delannoy makes the link $T_{x,y} = \binom{x}{x+y} - \binom{x-1}{x+y} = \frac{y-x+1}{y+1}\binom{x}{x+y}$ between entries from the rectangular array (our walks on $\mathbb{Z}$, here given by the binomial coefficients) and entries from the triangular array (walks constrained to remain in the upper plane, our Dyck paths). The numbers $T_{x,y}$ are called (in English) "ballot numbers", but they are also called Delannoy–Segner numbers in Albert Sade's review (in the Mathematical Reviews) of Touchard's article [87]. Kreweras [54] and Penaud [71] follow this terminology (quoting Riordan or Errera [38] but none of Delannoy's articles which all sank into oblivion). In conclusion, these "Delannoy–(Segner)" numbers $T_{x,y}$ are *not* the "famous" Delannoy numbers $D_{n,k}$ defined in Section 2.

4.2. **The length of the game (1888)** [26]**.** There are several contributions of Rouché and Bertrand in the Comptes Rendus de l'Académie des Sciences on the following problem that they call *the game*: "two players have n francs and play a game, at each round, the winner gets one franc from his opponent. One stops when one of the two players is ruined." When the game is fair, the probability to be ruined at the beginning of the round m is (with $q = \frac{m-n}{2}$):

$$\begin{aligned} &\frac{(-1)^{m-n}}{n}\sum_{k=1}^{n}(-1)^{k-1}\sin\Big(\frac{(2k-1)\pi}{2n}\Big)\cos^{m-1}\Big(\frac{(2k-1)\pi}{2n}\Big) \\ = \;&\frac{n}{2^{m-1}}\sum_{k=0}^{q/n}(-1)^k\frac{2k+1}{\frac{m+n}{2}+kn}\binom{m-1}{q-kn}. \end{aligned}$$

Rouché proves the left hand part with some determinant computations and Delannoy uses lattice paths to get the right hand part (claiming justly that there was a mistake in Rouché's first formula).

One can see this problem as a Dyck walk in the strip $[-n, n]$, that is why the formula is similar to the formula 14 from [23] in their enumeration of planted plane trees of bounded height (Feller [40] gives also some comments on this).

4.3. **How to use a chessboard to solve various probability theory problems (1889)** [27]**.** This is a potpourri of seven ballot-like or ruin-like problems partially solved by de Moivre, Laplace, Huyghens, Ampère, Rouché, Bertrand, André, ... for which Delannoy presents his simple solutions, obtained by his lattice paths enumeration method. He calls the lattice "chessboard". The different constraints corresponds to different kind of chessboards: triangular for walks in the upper-plane, rectangular for unconstrained walks, pentagonal for walks bounded from above, hexagonal for walks in a strip (modern authors from statistical physics sometimes talk about walks with a wall or two walls [53]). Delannoy numbers (and the two corresponding binomial formulae) appear at page 51. Delannoy says that it corresponds to the directed walk of a queen (sic), and that this problem was suggested to him by Laisant. This (and the further advertisement by Lucas of Delannoy's works, see e.g. p. 174 of [59] on "Delannoy's arithmetical square", which is exactly the array given in Section 2) answers to the question raised in our title. The authors who later wrote about Delannoy numbers/arrays then gave references to Lucas [59], whereas Delannoy's articles sank into oblivion.

4.4. **Various problems about the game (1890)** [29]**.** Using an enumeration argument, simplifying the sum that he obtained and then using the Stirling formula, he gives the asymptotic result $\frac{1}{\sqrt{2\pi}}\sqrt{2n}$ as the difference between the number of won and lost games, after $2n$ games. He also answers to other problems, e.g. what is the probability to have a group of $2, 3, \ldots, 8$ cards of the same color in a packet of 32 cards.

4.5. **Formulae related the binomial coefficients (1890)** [28]**.** He gives several binomial formulae, such as $\sum_{k=0}^{p}(p-2k)^2\binom{p}{k} = p2^p$.

4.6. **On the geometrical trees and their use in the theory of chemical compounds. (1894)** [30, 31]**.** A chemist asked for some explanations of Cayley's results, mentioned in a German review. Delannoy translated this review and corrected a computational mistake, giving his own method, without knowing [18, 19, 20]. This corresponds to the sequences `EIS 22` (centered hydrocarbons with n atoms) and `EIS 200` (bicentered hydrocarbons with n atoms). Application of combinatorics to enumeration of chemical configurations is a subject which will be later revisited by Pólya [72].

4.7. **How to use a chessboard to solve some probability theory problems (1895)** [34]**.** Delannoy makes a summary of 17 applications of his theory of triangular/square/pentagonal/hexagonal chessboard. The array of Delannoy numbers (see our Section 2) appears on the page 76 from this article.

4.8. **On a question of probabilities studied by d'Alembert (1895)** [33]**.** Delannoy corrects some mistakes in Montfort's solution to a problem raised by d'Alembert.

4.9. **A question of undetermined analysis (1897)** [35]**.** A review (by Professor Lampe) of this article can be found in the Jahrbuch über die Fortschritte der Mathematik. However, we were not able to get this article. There were in fact two journals whose name was "Journal de Mathématiques élémentaires" (one edited by Vuibert and the other edited by Bourget/Longchamps), neither of them seems to contain the quoted article.

4.10. **On the probability of simultaneous events (1898)** [36]**.** A priest wrote an article in which he was bravely contesting the "third Laplace principle" $P(A \cap B) = P(A)P(B)$ for two independent events, arguing with three examples. Delannoy shows that they present a misunderstanding of "independent events", which goes back to the original fuzzy definition by de Moivre.

4.11. **Contributions to "L'Intermédiaire des Mathématiciens"** [32]**.** This journal was created in 1894 by C.-A. Laisant and Émile Lemoine. It is quite similar to the actual sci.math newsgroups. This journal was indeed only made of problems/questions/solutions/answers.

During the quoted period, numerous famous mathematicians made some contributions to this journal: Appell, Borel, Brocard, Burali-Forti, Cantor, Catalan, Cayley, Cesàro, Chebyshev, Darboux, Dickson, Goursat, Hadamard, Hermite, Jumbert, Hurwitz, Jensen, Jordan, Kempe, Koenigs, Laisant, Landau, Laurent, Lemoine, Lerch, Lévy, Lindelöf, Lipschitz, Moore, Nobel, Picard, Rouché...

From 1894 until 1908 (date of his last mathematical contribution), Delannoy was an active collaborator: he raised or solved around 70 questions/problems. These are questions number 20, 29, 32, 51, 84, 95, 138, 139, 140, 141, 142, 155, 191, 192, 314, 330, 360, 371, 407, 424, 425, 443, 444, 451, 453, 493, 494, 514, 601, 602, 603, 664, 668, 749, 1090, 1304, 1360, 1459, 1471, 1479, 1551, 1552, 1578, 1659, 1723, 1869, 1875, 1894, 1922, 1925, 1926, 1938, 1939, 2074, 2076, 2077, 2091, 2195, 2212, 2216, 2251, 2305, 2325, 2452, 2455, 2583, 2638, 2648, 2868, 2873, 3326.

These contributions can be classified in three sets: the problems and solutions related to combinatorics (enumeration and applications to probabilistic problems), problems and solutions related to elementary number theory (representations of integers as sum of some powers, Fermat-like problems), and questions/answers related to Lucas' books (so mainly recreative mathematics, but not so trivial problems as it includes, *e.g.*, the four color problem).

To these articles, perhaps one should add some *récréations* of [61] (compare the warning in its preface), and also some problems written by Lucas, but with Delannoy's solutions. The same holds for articles written by Lucas in *La Nature*.

Finally, there are some books [16, 22, 44, 58, 59, 61, 62, 63] (Lucas, Frolow, and Catalan intensively corresponded with Delannoy for their books) or articles [2, 12, 14, 5, 49, 50, 51, 54, 65, 69, 70, 71, 74, 75, 78, 83, 87, 88, 89] which mention either Delannoy numbers or some of Delannoy's results/methods.

5. Other Delannoy's works

Besides mathematics, Delannoy painted watercolors and, perhaps more importantly, studied history. Indeed, from 1897 to 1914, he published 29 accurate archaeological/historical articles in the Mémoires de la Société des Sciences Naturelles et archéologiques de la Creuse.

Let us give a taste of Delannoy's writer talent: here some titles of his articles: "On the signification of word *ieuru*", "One more word about *ieuru*", "A riot in Guéret in 1705", "Aubusson's tapestries", "A bigamist in Guéret", "Grapevines in the Creuse", a lot of studies "Criminal trials in the Marche. The case . . . ", several studies on abbeys and some "Critical list of the abbots from . . . ", and last but not least, "An impotence trial in the 18th century". When he died, at the age of 81, about a dozen other articles were still in progress.

Delannoy is surely one of the last "self-made" mathematicians who succeeded in getting a name in this field, rivaling professional mathematicians. What he discovered is nowadays well understood and can be classified as "basic enumerative combinatorics". However, despite the simplicity of his tools, it seems to us that Delannoy's work (and more generally, the underlying combinatorics) is a nice example of what could, but is actually not taught to young students (or even in high-schools), as an introduction to research in mathematics, also allowing the use of computers and computer algebra softwares. This kind of mathematics is only present at the mathematical Olympiads. This attractive bridge between enumeration, geometry, probability theory, analysis, . . . deserves a better place.

It appears very clearly, thanks to the archives of the Society of Natural Sciences and Archaeology, that besides his own publications, Delannoy played a great rôle in checking proofs for numerous mathematicians and historians who wrote to have his contribution [4]. The archive from the Society and from Delannoy's family in Guéret reveals a true *honnête homme*, as defined in the seventeenth century.

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The first author's interest to Delannoy numbers comes from a talk that Marko Petkovšek gave in the Algorithms Seminar at INRIA in 1999 (a summary of this talk can be found in [8]). As an example, he was dealing with chess king moves (his general result about the nature of different multidimensional recurrences can be found in the article [13]). M. Petkovšek asked the first author what he knew about Delannoy, and C. Banderier then started to conduct an investigation...

The second author's interest to Delannoy numbers comes from her own works. As a researcher in Temporal Representation and Reasoning, she developed a model based on formal languages theory [77] instead of the logical or relational approaches. This framework allowed her to enumerate easily all possible temporal relations between n independent events-chronologies. Then she tried to know if some of these sequences were already known. In the $n = 2$ case, Sloane's On-Line EIS provided her the name of Delannoy. She asked everybody she met who is Delannoy? She already started her own investigation when Philippe Flajolet sent her to the first author.

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[9] `http://perso.wanadoo.fr/jp-l/SSC23/`
[10] `http://www.emis.de/projects/JFM/`
[11] `http://www.mpi-sb.mpg.de/services/library/`
[12] `http://cnum.cnam.fr/`
[13] `http://www-rocq.inria.fr/doc/`
[14] `http://bu.univ-lemans.fr/`
[15] http://www.univ-paris13.fr/
[16] `http://www.cirm.univ-mrs.fr/SitBib/Bibli/debut.html`
[17] `http://bleuet.bius.jussieu.fr/`

Although very long, the following bibliography is not a complete list of the so vast literature on lattice paths, but it is mainly a tentative bibliography (by no way exhaustive) about enumeration of lattice paths "related" to Delannoy lattice paths and other Delannoy works.

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